1. Lecture 1 (Jan 13): Affine algebraic sets

Course info

- shelter 8 (building across the street; follow the crowd)
- course website: https://achinger.impan.pl/ag2026.html
- email: pachinger@impan.pl
- Moodle, Slack, Zoom, Google Drive
- weekly homework posted on the course website, submit solutions by email with filename `Lastname-N.pdf` where N is the number of the problem set
- extra credit: problems marked with an asterisk (due end of term), short (3–5 pages) term papers on a topic of your choice (more info soon)
- reading week (no class): Apr 6–10, exam (written or oral) in the last week (Apr 14 or 16)
- office hours: Thursdays 10am (email me if you plan to come)
- I might post some lecture notes (in fact, I am just doing that)
- literature (see Google Drive):
  1. G. Kempf *Algebraic Varieties* (excellent for a one semester course)
  2. R. Hartshorne *Algebraic Geometry* (a bit heavy for us, but we will read parts of it)
  3. R. Vakil *The Rising Sea: Foundations of Algebraic Geometry*
  4. M. Reid *Undergraduate Algebraic Geometry*
  5. D. Mumford *The Red Book of Varieties and Schemes*

1.1. Affine algebraic sets and their k-points

Recommended reading for this lecture: Hartshorne, I.1 (and bits of I.2).

Algebraic geometry studies algebraically defined geometric objects, of which the most basic are **(affine) algebraic sets**. Fix a field k and consider a system of polynomial equations in n variables

$$X : \begin{cases} f_1(T_1,\ldots,T_n) = 0 \\ \quad \ldots \\ f_r(T_1,\ldots,T_n) = 0 \end{cases} \tag{1.1.1}$$

where $f_1,\ldots,f_r \in k[T_1,\ldots,T_n]$. Importantly, our basic object X is this system, not its **set of solutions** in k^n, which we denote by $X(k)$:

$$X(k) = \{(x_1,\ldots,x_n) \in k^n \,:\, f_i(x_1,\ldots,x_n) = 0 \text{ for } i = 1,\ldots,r\} \subseteq k^n.$$

More generally, if K is a field containing k (or just a k-algebra), we can define $X(K) \subseteq K^n$ as the set of solutions of (1.1.1) in K^n. We also define the **coordinate ring** of X as the k-algebra

$$A = \mathcal{O}(X) = k[T_1, \dots, T_n]/(f_1, \dots, f_r).$$

Examples 1.1.1. The most basic examples of systems of polynomial equations:

(a) If $r = 0$ (i.e. n variables and no equations), we call the system the **affine n-space** and denote it by $\mathbb{A}^n$. We have $\mathbb{A}^n(K) = K^n$ and $\mathcal{O}(\mathbb{A}^n) = k[T_1, \dots, T_n]$. For a system X with n variables, we write $X \subseteq \mathbb{A}^n$ to signify that $X(K) \subseteq K^n$.

(b) A **hypersurface** is the system consisting of a single equation $f(T_1, \dots, T_n) = 0$, where $f \in k[T_1, \dots, T_n]$ is a non-constant polynomial (often assumed to be irreducible). If $\deg(f) = 1, 2, 3, 4, 5, 6, 7, 8$ we call f a hyperplane, a quadric, a cubic, a quartic, a quintic, a sextic, a septic, an octic.

(c) A **plane curve** is a hypersurface in $\mathbb{A}^2$, i.e. a system C with a single equation

$$f(X, Y) = 0$$

where $f \in k[X, Y]$ is a non-constant polynomial. For example, the *lemniscate of Bernoulli* is defined by the equation

$$(X^2 + Y^2)^2 = X^2 - Y^2.$$

(d) If $\deg(f) = 2$ and f is irreducible, we call C a **conic**. That is, a conic is a quadric hypersurface in $\mathbb{A}^2$.

(e) Consider $k = \mathbb{R}$ and the conic C defined by

$$X^2 + Y^2 = -1.$$

We have $C(\mathbb{R}) = \emptyset$. However, $\mathcal{O}(C) \neq 0$ and $C(\mathbb{C}) \neq \emptyset$.

(f) The "fat point" $X \subseteq \mathbb{A}^1$ defined by the single equation

$$T^2 = 0.$$

We have $\mathcal{O}(X) = k[T]/(T^2)$, which is non-reduced (has a nonzero nilpotent element, namely T). For any field K, we have $X(K) = \{0\}$, which is the same as for the equation $X' : T = 0$. We need more advanced technology (schemes) to distinguish between the geometric objects X and X'. For now though we shall mostly stick to systems of equations which give reduced k-algebras.

We note that the algebra $A = \mathcal{O}(X)$ "remembers" the set $X(K)$, namely we have a bijection

$$X(K) = \mathrm{Hom}_k(A, K)$$

between the set of K-valued solutions and the set of k-algebra homomorphisms from A to K. Indeed, giving a map $\phi\colon k[T_1,\dots,T_n] \to K$ is the same as giving its values $x_i = \phi(T_i)$ i.e. an element $(x_1,\dots,x_n) \in K^n$. Such a map factors (uniquely) through $A = k[T_1,\dots,T_n]/(f_1,\dots,f_r)$ if and only if $\phi(f_j) = 0$. But

$$\phi(f_j(T_1,\dots,T_n)) = f_j(\phi(T_1),\dots,\phi(T_n)) = f_j(x_1,\dots,x_n)$$

which happens precisely when $(x_1,\dots,x_n) \in X(K)$.

Thus, a system X (1.1.1) determines A which determines the solution set $X(k)$. We regard X as too rigid, the set $X(k)$ as too primitive for describing a geometric object (for example, it could be empty), and the algebra A as just right.

The distinction between X, A, and $X(k)$ becomes less serious when k is algebraically closed, and indeed most of the methods of algebraic geometry are developed over $k = \bar{k}$, even if the motivation is the study of $X(k)$ for $k = \mathbb{R}$, $\mathbb{Q}$, or a finite field. Life is easier over an algebraically closed field k thanks to Hilbert's Nullstellensatz, which says that the only reason for the system (1.1.1) to have no solutions in k^n is that we can algebraically rearrange the equations to obtain the equation $1 = 0$:

Theorem 1.1.2 (Hilbert's Nullstellensatz)**.** *Consider a system of polynomial equations* (1.1.1) *over a field k. The following are equivalent:*

(a) $X(K) = \emptyset$ *for every field K containing k;*

(b) $X(\bar{k}) = \emptyset$*, where $\bar{k}$ is an algebraic closure of k;*

(c) there exist polynomials $h_1,\dots,h_r \in k[T_1,\dots,T_n]$ such that

$$1 = h_1 f_1 + \dots + h_r f_r.$$

(d) $\mathcal{O}(X) = 0$.

Proof. The implications (a)⇒(b), (c)⇔(d)⇒(a) are obvious. We shall deduce the remaining (b)⇒(d) from the following theorem, another version of Nullstellensatz.

Theorem 1.1.3 (Basic form of Nullstellensatz)**.** *Let k be a field and let K be a finitely generated k-algebra which is a field. Then K is a finite extension of k.*

Last term, which we proved it using the Artin–Tate lemma. Just for fun, let us give a simple proof in case k is uncountable (e.g. $k = \mathbb{C}$). Note first that it suffices to show that every $x \in K$ is algebraic over k (as a finitely generated algebraic extension is finite). Suppose $x \in K$ is not algebraic over k, then we have an injection $k(T) \subseteq K$. Now, we get a contradiction because

- $k(T)$ has uncountable dimension over k, as the elements $1/(T-\alpha)$ are linearly independent over k for $\alpha \in k$ and k is uncountable;
- K has countable dimension over k, since being finitely generated over k it is a quotient of $k[T_1,\dots,T_n]$ for some $n \geq 0$, and this space has a countable basis consisting of all monomials in the T_i.

To show (b)⇒(d), suppose $A = \mathcal{O}(X) \neq 0$. Thus A has a maximal ideal $\mathfrak{m}$, and $K = A/\mathfrak{m}$ is a field which is generated over k by images of the T_i. By the above theorem, it is finite over k, and hence we can find an embedding $K \subseteq \bar{k}$, so that $X(K) \subseteq X(\bar{k})$. Now, by construction $X(K) = \mathrm{Hom}_k(A, K)$ is non-empty (because we have the quotient map $A \to A/\mathfrak{m} = K$), and hence $X(\bar{k})$ is non-empty. □

From now on we shall assume that k is algebraically closed.

This assumption implies in particular that the set $X(k) = \mathrm{Hom}_k(A, k)$ coincides with the set $\mathrm{MSpec}(A)$ of maximal ideals of A. I will also use $\mathbb{A}^n$ to mean $\mathbb{A}^n(k) = k^n$.

Definition 1.1.4. Let $P = k[T_1, \dots, T_n]$ be the polynomial ring.

(1) For $f \in P$, we write

$$V(f) = \{(x_1, \dots, x_n) \in k^n \,:\, f(x_1, \dots, x_n) = 0\}$$

and $D(f) = k^n \setminus V(f)$.

(2) For an ideal $I \subseteq P$, we write $V(I) = \bigcap_{f \in I} V(f)$. (Note that we do not define $D(I)$.)

(3) An **affine algebraic set** is a subset $Z \subseteq k^n$ of the form $V(I)$ for some $I \subseteq P$.

(4) For a subset $Z \subseteq k^n$, we denote by $\mathcal{I}(Z) \subseteq P$ the ideal

$$\mathcal{I}(Z) = \{f \in P \,:\, f(x_1, \dots, x_n) = 0 \text{ for all } x \in Z\}.$$

Note that for $f_1, \dots, f_r \in P$ and $I = (f_1, \dots, f_r)$ we have $V(I) = V(f_1) \cap \dots \cap V(f_r)$.

Proposition 1.1.5. *Let $P = k[T_1, \dots, T_n]$ be the polynomial ring.*

(a) For any family of ideals $I_\alpha \subseteq P$ we have $\bigcap V(I_\alpha) = V(\sum I_\alpha)$.

(b) For two ideals $I, J \subseteq P$ we have $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.

(c) Affine algebraic sets are the closed sets for a topology on k^n (called the Zariski topology).

(d) The open sets $D(f)$ form a base for the topology on k^n closed under pairwise intersection: $D(f) \cap D(g) = D(fg)$.

(e) For every subset $Z \subseteq k^n$, the ideal $\mathcal{I}(Z)$ is radical and $V(\mathcal{I}(Z)) = \overline{Z}$ (the closure of Z).

(f) For every ideal $I \subseteq k^n$ we have $\mathcal{I}(V(I) = \sqrt{I}$ (the radical of I).

(g) The maps V and $\mathcal{I}$ establish mutually inverse bijections between closed subsets of k^n and radical ideals of P.

Proof. Everything is easy to show except for the containment $\mathcal{I}(V(I)) \subseteq \sqrt{I}$ in (f), another form of the Nullstellensatz. Let us deduce it from Theorem 1.1.2. Write $I = (f_1, \dots, f_r)$. We must show that if $g \in P$ vanishes on $V(f_1, \dots, f_r)$ then $g^n \in I$ for some $n \geq 1$. Equivalently, g is nilpotent in $A = P/I$. Consider the system of equations in $n+1$ variables $T_1, \dots, T_n, T_{n+1}$:

$$X' = \begin{cases} f_1(T_1, \dots, T_n) = 0 \\ \dots \\ f_r(T_1, \dots, T_n) = 0 \\ g(T_1, \dots, T_n) \cdot T_{n+1} - 1 = 0. \end{cases}$$

Then the assumption $V(g) \supseteq V(f_1, \dots, f_r)$ is equivalent to $X'(k) = \emptyset$. Thus by Theorem 1.1.2 we have $A' = 0$ where $A' = \mathcal{O}(X')$. However

$$A' = \left(\frac{k[T_1, \dots, T_n]}{(f_1, \dots, f_r)} \right) [T_{n+1}]/(gT_{n+1} - 1) = A[g^{-1}],$$

the localization of A at g. Then $A' = 0$ means that $0/1 = 1/1$ in $A[g^{-1}]$, which by definition of localization means that $g^n = 0$ in A, and we are done. □

Note that the last result implies that we can recover A up to nilpotents from the set $X(k) \subseteq k^n$:

Corollary 1.1.6. *We have* $A/\sqrt{0} \simeq P/\mathcal{I}(X(k))$.

Examples 1.1.7. (a) A proper subset of $\mathbb{A}^1 = k$ is closed if and only if it is finite.

(b) Let $Z \subseteq \mathbb{A}^2$ be a proper closed subset. Then Z is the union of a plane curve $f(X,Y) = 0$ and a finite set. (We will prove this later.)

1.2. Projective algebraic sets

The multiplicative group $k^\times$ acts freely on the open subset $k^{n+1} \setminus 0$ of k^{n+1} by

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n).$$

We define the **projective n-space** to be the quotient (orbit space)

$$\mathbb{P}^n(k) = (k^{n+1} \setminus 0)/k^\times.$$

We denote by $\pi\colon k^{n+1} \setminus 0 \to \mathbb{P}^n(k)$ the quotient map, and by $(x_0 : \ldots : x_n)$ the image of $(x_0, \dots, x_n) \in k^{n+1} \setminus 0$.

Let $P = k[T_0, \dots, T_n]$. For $(x_0 : \dots : x_n) \in \mathbb{P}^n(k)$, the expression

$$f(x_0, \dots x_n)$$

does not make sense. However, if f is *homogeneous*, the condition

$$f(x_0, \dots x_n) = 0$$

makes sense (i.e. is independent of the choice of a representative $(x_0,\dots,x_n)$). Recall that a polynomial $f \in P$ is **homogeneous of degree** d if we have an equality of polynomials in $P[\lambda]$

$$f(\lambda T_0,\dots,\lambda T_n) = \lambda^d f(T_0,\dots,T_n),$$

or equivalently if all monomials in f are of the same degree d. We say that f is homogeneous if it is homogeneous of degree d for some $d \geq 0$. If $P_d \subseteq P$ is the subspace of homogeneous polynomials of degree d, then

$$P = \bigoplus_{d\geq 0} P_d.$$

An ideal $I \subseteq P$ is **homogeneous** if it is generated by a set of homogeneous elements, or equivalently if

$$I = \bigoplus_{d\geq 0} (I \cap P_d).$$

Lemma 1.2.1. *Let $Z \subseteq k^{n+1}$ be a closed subset. The following are equivalent:*

(a) Z is **conical***, i.e. invariant under the action of $k^\times$;*

(b) $Z = V(I)$ for a homogeneous ideal $I \subseteq P$;

(c) the ideal $\mathcal{I}(Z) \subseteq P$ is homogeneous.

Proof. The implications (c)⇒(b)⇒(a) are straightforward. To show (a)⇒(c), let $f \in \mathcal{I}(Z)$ and write $f = \sum f_d$ where $f_d \in P_d$. We must show that all f_d belong to $\mathcal{I}(Z)$ i.e. vanish on Z. Let $(x_0,\dots,x_n) \in Z \setminus 0$, and consider the function $\phi\colon k \to k$ given by

$$\phi(\lambda) = f(\lambda x_0,\dots,\lambda x_n) = \sum_{d\geq 0} f_d(\lambda x_0,\dots,\lambda x_n) = \sum_{d\geq 0} f_d(x_0,\dots,x_n)\lambda^d.$$

Since Z is conical, we have $(\lambda x_0,\dots,\lambda x_n) \in Z$, and hence $\phi(\lambda)$ is identically zero. Since it is a polynomial in λ by the above expression, all of its coefficients are zero, so $f_d(x_0,\dots,x_n) = 0$ for all d. □

Definition 1.2.2. For a homogeneous ideal $I \subseteq P$, let

$$V_{\mathbb{P}}(I) = \{(x_0 : \dots : x_n) \,:\, f(x_0,\dots,x_n) = 0 \text{ for all homogeneous } f \in I\} = \pi(V(I)\setminus 0) \subseteq \mathbb{P}^n(k).$$

A **projective algebraic set** is a subset of $\mathbb{P}^n(k)$ of the form $V_{\mathbb{P}}(I)$ for some homogeneous ideal $I \subseteq P$.

As in the affine case, projective algebraic sets are the closed sets of a topology on $\mathbb{P}^n(k)$ called the **Zariski topology**, with basis of open sets given by

$$D_{\mathbb{P}}(f) = \{(x_0 : \dots : x_n) \,:\, f(x_0,\dots,x_n) \neq 0\} = \pi(D(f))$$

for homogeneous $f \in P$.

Let us show how $\mathbb{P}^n(k)$ can be expressed as the union of $n+1$ copies of $\mathbb{A}^n$. Let

$$U_i = D_{\mathbb{P}}(T_i) = \{(x_0 : \cdots : x_n) : x_i \neq 0\} \subseteq \mathbb{P}^n(k)$$

$$V_i = \{(x_0, \ldots, x_n) \in k^{n+1} : x_i = 1\} \subseteq k^{n+1} \setminus 0.$$

Then $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$ and for each i, the restriction of π to V_i gives a homeomorphism $V_i \simeq U_i$, with inverse given by

$$(x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i} : \cdots : \frac{x_n}{x_i} \right).$$

Example 1.2.3 (Projective line)**.** Let us explicate the above description for $n = 1$. The projective line $\mathbb{P}^1$ is the union of $U_0 = 1 \times k \simeq \mathbb{A}^1$ with coordinate x_1 and $U_1 = k \times 1 \simeq \mathbb{A}^1$ with coordinate x_0. We have $U_0 \cap U_1 \simeq k^\times$ with coordinate $x_1 = x_0^{-1}$. We can write

$$\mathbb{P}^1(k) = \mathbb{A}^1 \sqcup \{\infty\}, \qquad \infty = (0:1).$$

Example 1.2.4 (Projective plane)**.** Consider $n = 2$ and let us name the coordinates T_0, T_1, T_2 by X, Y, Z. Let $U = U_2 = \{Z \neq 1\} = \mathbb{A}^2$ with coordinates $x = X/Z$ and $y = Y/Z$. Then $\mathbb{P}^2 \setminus U = V(Z)$ can be identified with $\mathbb{P}^1$ with homogeneous coordinates $(x : y)$.

The following lemma allows us to compute the projective closure of an affine hypersurface. For a polynomial $f \in k[T_1, \ldots, T_n]$ (no T_0) of degree $d \geq 0$, write $f = \sum_{e=0}^d f_d$ where f_e is homogeneous of degree e, and let us define its **homogenization** as

$$\bar{f} = \sum_{e=0}^{d} T_0^{d-e} f_e \quad \in P_d.$$

This is the unique homogeneous polynomial of degree d satisfying

$$f(T_1, \ldots, T_n) = \bar{f}(1, T_1, \ldots, T_n).$$

For example,

$$f = T_1^2 - T_2^3 - T_2 \quad \Rightarrow \quad \bar{f} = T_0 T_1^2 - T_2^3 - T_0^2 T_2.$$

Lemma 1.2.5. *Let $f \in k[T_1, \ldots, T_n]$ be a nonzero polynomial with homogenization $\bar{f} \in k[T_0, \ldots, T_n]$. Then the closure of $V(f) \subseteq U_0 = k^n$ in $\mathbb{P}^n(k)$ is given by $V_{\mathbb{P}}(\bar{f})$.*

We didn't prove this lemma on Jan 13. We give a proof in the subsequent subsection.

1.3. Problem session

We discussed charts on the projective space. We also introduced the following notion (used in problem 6).

Definition 1.3.1. Let $f \in k[X, Y]$ be a square-free nonconstant polynomial, defining an affine curve $C = V(f) \subseteq k^2$. We say that a point $P = (x, y) \in C$ is **singular** if $\partial f / \partial X$ and $\partial f / \partial Y$ both vanish at P. Otherwise, we say that P is a **nonsingular** or **smooth** point of C.

We mentioned, but didn't prove, the fact that every curve has only finitely many singular points.

List of problems:

(a) *Show (a)⇒(c) in Lemma 1.2.1*

Solution. See the proof of Lemma 1.2.1.

(b) *Find a non-homogeneous ideal $I \subseteq k[T_0, \dots, T_n]$ whose zero set $Z = V(I) \subseteq k^{n+1}$ is conical.*

Solution. Consider $I = (X^2 + Y, Y^2) \subseteq k[X,Y]$. This ideal is not homogeneous since $X^2 + Y \in I$ but $Y \notin I$. However, its zero set is $\{(0,0)\}$ (note $\sqrt{I} = (X,Y)$), which is conical.

(c) *Find the points at infinity of the affine plane curves $Y = X^2$ and $XY = 1$.*

Solution. Assuming Lemma 1.2.5 (problem 5 below), the closures of $V(Y - X^2)$ and $V(XY - 1)$ are cut out by the homogenized equations $YZ = X^2$ and $XY = Z^2$. Setting $Z = 0$ we get $0 = X^2$ and $XY = 0$. Thus, the points at infinity are $(0 : 1 : 0)$ (with multiplicity two) in the first example and $\{(0 : 1 : 0), (1 : 0 : 0)\}$ (corresponding to the vertical and horizontal asymptote) in the second example.

(d) *Prove that the Zariski topology on $k^2 = k \times k$ is not the product topology (with both factors given the Zariski topology).*

Solution. The diagonal $V(X - Y) \subseteq k^2$ is closed in the Zariski topology, but not in the product topology. In fact, a proper subset $Z \subsetneq k^2$ is closed in the product topology if and only if it is a finite union of horizontal lines, vertical lines, and points (straightforward, details omitted).

Klymentii asked if we can show that the two spaces are not homeomorphic (possibly by a map which is not the identity). The answer seems to be no, but to handle this we need to know something about irreducible closed subsets (next lecture). More precisely, irreducible closed subsets of $k \times k$ are precisely the vertical lines, horizontal lines, points, and the entire space. Thus the intersection of two distinct proper irreducible closed subsets has at most one point. However, in k^2 the subsets $V(Y)$ and $V(Y - X(X-1))$ are irreducible and have exactly two points in common.

Bonus question: show that $\mathbb{A}^2$ and $\mathbb{P}^2$ are not homeomorphic. See also Problem 6* on Problem Set 1.

(e) *Prove Lemma 1.2.5*

We didn't solve this problem. I added a proof in the next subsection.

(f) *(Hartshorne (I, Ex. 5.1)* Find the singular points of the following curves (assuming $\mathrm{char}(k) \neq 2$).

(a) $X^2 = X^4 + Y^4$;

(b) $XY = X^6 + Y^6$;

(c) $X^3 = Y^2 + X^4 + Y^4$;

(d) $X^2Y + XY^2 = X^4 + Y^4$.

See Hartshorne's book for pictures of these singularities.

Solution. (c) Hands-on computation gives that $(0,0)$ is the only singular point.

1.4. Bonus: Computing the closure

The following lemma allows us to compute the closure of a locally closed subset of $\mathbb{A}^n$.

Lemma 1.4.1. *Let $I \subseteq P = k[T_1, \dots, T_n]$ be an ideal and let $g \in P$. Let*

$$W = V(I) \subseteq \mathbb{A}^n \quad \textit{and} \quad U = D(g) \subseteq \mathbb{A}^n,$$

and let $W' \subseteq \mathbb{A}^n$ be the closure of $W \cap U$. Consider the ideal

$$I' = \ker(P \to (P/I)[g^{-1}]) = \{f \in P : g^n f \in I \textit{ for some } n \geq 1\}.$$

Then $W' = V(I')$. Moreover, if I is radical, then so is I'.

Proof. First, we show the equality between the two given definitions of I'. This follows from the general fact that for an element g of a ring B the kernel of the localization $B \to B[g^{-1}]$ consists of all $f \in B$ such that $g^n f = 0$ for some $n \geq 0$.

Now, the closure of $W \cap U$ is the intersection of all $V(f)$ for $f \in P$ which vanish on $W \cap U$. Let $f \in P$ be such an element. As in the proof of the variant of Nullstellensatz in Proposition 1.1.5, we consider the zero set

$$W'' = V(I, gT_{n+1} - 1) \subseteq \mathbb{A}^{n+1}.$$

The corresponding coordinate ring is

$$\mathcal{O}(W'') = k[T_1, \dots, T_n, T_{n+1}]/(I, gT_{n+1} - 1) = (P/I)[g^{-1}].$$

Then f (treated as an element of $P[T_{n+1}] = k[T_1, \dots, T_n, T_{n+1}]$) vanishes on W'', and hence (by Proposition 1.1.5(f)) its image in $\mathcal{O}(W'') = (P/I)[g^{-1}]$ is nilpotent. This means that $f^m g^n \in I$ for some $n \gg 0$, in other words $f \in \sqrt{I'}$. Thus $\overline{W \cap U} = V(\sqrt{I'}) = V(I')$, and we are done. □

The ideal I' in Lemma 1.4.1 is sometimes called the **g-saturation** of I. Using Lemma 1.4.1 we can now give a proof of Lemma 1.2.5.

Proof of Lemma 1.2.5. Let W be the closure of $V(f)$ and let $V = V_{\mathbb{P}}(\bar{f})$. We want to show that $W = V$, and since $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$, it suffices to check that $W \cap U_i = V \cap U_i$ for each i. For $i = 0$ this is clear, as $W \cap U_0 = V(f) = V \cap U_0$. Permuting the variables $T_1, \dots, T_n$, without loss of generality it suffices to consider the case $i = n$ (which saves us a minor annoyance with indices). The set U_n has coordinates $(t_0, \dots, t_{n-1})$ where $t_i = T_i/T_n$, and $U_0 \cap U_n$ is the set $D(t_0)$. Then $V(f) \cap U_n$ is

$$\{(t_0, \dots, t_{n-1}) : t_0 \neq 0 \,\&\, f(t_1/t_0, \dots, t_{n-1}/t_0, 1/t_0) = 0\}.$$

Note that

$$t_0^d f(t_1/t_0, \ldots, t_{n-1}/t_0, 1/t_0) = \bar{f}(t_0, t_1, \ldots, t_{n-1}, 1).$$

Call this element $f' = \bar{f}(t_0, t_1, \ldots, t_{n-1}, 1) \in k[t_0, \ldots, t_{n-1}]$.

Now, by Lemma 1.4.1 the closure of $V(f) \cap U_n$ in $U_n = \mathbb{A}^n$ is cut out by the t_0-saturation of the ideal (f'). Thus we must show this ideal is t_0-saturated. Note that by construction,

$$f' = \bar{f}(t_0, t_1, \ldots, t_{n-1}, 1) = f_d(t_1, \ldots t_{n-1}, 1) + t_0 \sum_{e<d} t_0^{d-e-1} f_e(t_1, \ldots, t_{n-1}, 1)$$

is not divisible by t_0. Let $g \in k[t_0, \ldots, t_{n-1}]$ and suppose that f' divides $t_0^n g$. Since t_0 does not divide f' and the polynomial ring $k[t_0, \ldots, t_{n-1}]$ is a UFD, we deduce that f' divides g. □