19. Lecture 19 (Apr 14): More projective tools

Our next big goal is to develop intersection theory on surfaces. For this, we need two more tools: the notion of an ample line bundle and the Bertini theorem on hyperplane sections. $\mathcal{A}$

Recommended reading: Hartshorne II 7 and V 1, Kempf §10.9.

19.1. Line bundles and maps to projective space

Let $\mathcal{F}$ be a coherent sheaf on a projective variety X. Then the vector space $\Gamma(X,\mathcal{F})$ is finite-dimensional, and we can make it into a free $\mathcal{O}_X$-module $\Gamma(X,\mathcal{F})\otimes_k \mathcal{O}_X$. We have a canonical map

$$\eta : \Gamma(X,\mathcal{F})\otimes_k \mathcal{O}_X \longrightarrow \mathcal{F} \tag{19.1.1}$$

which on an open $U \subseteq X$ is the map

$$s\otimes f \mapsto f\cdot(s|_U) \quad : \quad \Gamma(X,\mathcal{F})\otimes_k \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U).$$

(If we denote by p the unique map $X \to \star$ to the point, then this map is simply the counit $p^*p_*\mathcal{F} \to \mathcal{F}$.)

Definition 19.1.1. We say that a coherent sheaf $\mathcal{F}$ on a projective variety X is **globally generated** if the map (19.1.1) is surjective.

We note right away that $\mathcal{F}$ is globally generated if and only if it is a quotient of $\mathcal{O}_X^n$ for some $n \geq 0$. In particular, if $\mathcal{F}$ is globally generated and $\mathcal{F} \to \mathcal{F}'$ is surjective, then $\mathcal{F}'$ is globally generated.

Examples 19.1.2.

1. A free sheaf $\mathcal{O}_X^n$ is globally generated;
2. For every closed $Z \subseteq X$, the sheaf $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$ is globally generated.
3. If $\mathcal{F}$ and $\mathcal{F}'$ are globally generated, then so is $\mathcal{F}\otimes\mathcal{F}'$. This follows from right-exactness of $\otimes$.
4. An invertible sheaf $\mathcal{L}$ is globally generated if and only if for every $x \in X$ there exits an $s \in H^0(X,\mathcal{L})$ with $s(x) \neq 0$. A point $x \in X$ such that $s(x) = 0$ for all $s \in H^0(X,\mathcal{L})$ is called a base point, and we sometimes call a globally generated invertible sheaf **base-point free**.
5. For $\mathcal{L} = \mathcal{O}(1)$ on $X = \mathbb{P}^n$, the map (19.1.1) is
$$(x_0,\dots,x_n)\colon \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \tag{19.1.2}$$
(as we have seen already, it is surjective, with kernel $\Omega^1_{\mathbb{P}^n}(1)$).
6. Serre's theorem (Lecture 16, Theorem 16.2.5 and Remark 16.2.6) is equivalent to the statement that for every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ the sheaf $\mathcal{F}(d) := \mathcal{F}\otimes\mathcal{O}(d)$ is globally generated for $d \gg 0$.

The following lemma describes the universal property of $\mathbb{P}^n$. In its statement we use the following terminology: for a coherent sheaf $\mathcal{F}$ on a variety X, an **invertible sheaf quotient** of $\mathcal{F}$ is a surjection $\mathcal{F} \to \mathcal{L}$ onto an invertible sheaf $\mathcal{L}$, where we deem $\phi\colon \mathcal{F} \to \mathcal{L}$ and $\phi'\colon \mathcal{F} \to \mathcal{L}'$ equal if $\ker(\phi) = \ker(\phi')$ (or equivalently if $\mathcal{L} \simeq \mathcal{L}'$ under $\mathcal{F}$). With this convention, invertible sheaf quotients of $\mathcal{F}$ form a set. Equation (19.1.2) exhibits a particular invertible sheaf quotient of $\mathcal{F} = \mathcal{O}_X^{n+1}$ on $X = \mathbb{P}^n$. For any map $f\colon Y \to X$, the right-exactness of f^* shows that f^* maps invertible sheaf quotients of $\mathcal{F}$ to invertible sheaf quotients of $f^*\mathcal{F}$.

Lemma 19.1.3. *For any algebraic set (or scheme) X, the map*

$$\operatorname{Hom}(X,\mathbb{P}^n) \xrightarrow{\simeq} \{\textit{invertible sheaf quotients of } \mathcal{O}_X^{n+1}\}, \qquad (f\colon X\to\mathbb{P}^n)\mapsto f^*((19.1.2)).$$

is bijective.

Before giving the proof, let us discuss the intuition behind this result. For this, it is better to formulate it in a coordinate-free manner. Let $V\simeq k^{n+1}$ be a k-vector space of dimension $n+1$. Then

$$\mathbb{P}V = (V\setminus\{0\})/k^\times \simeq \mathbb{P}^n$$

parametrizes lines (one-dimensional linear subspaces) in V. We have $\Gamma(\mathbb{P}V,\mathcal{O}(1)) = V^\vee$, and $\mathbb{P}V^\vee$ parametrizes hyperplanes in V, or equivalently one-dimensional quotients of V. A correctly defined "family of one-dimensional quotients of V parametrized by X" is precisely an invertible sheaf quotient of the free $\mathcal{O}_X$-module $V\otimes_k\mathcal{O}_X$, and thus such quotients should be in bijection with maps $X\to\mathbb{P}V^\vee$.

Proof. Let us describe the inverse map (details left to the reader). Let $s=(s_0,\ldots,s_n)\colon\mathcal{O}_X^{n+1}\to\mathcal{L}$ be a surjective map, where $s_i\in\Gamma(X,\mathcal{L})$. If $U_i = D(s_i) = \{x\in X : s_i(x)\neq 0\}$, then surjectivity of s means that $X=\bigcup U_i$. On U_i, the section s_i trivializes $\mathcal{L}$, meaning that $s_i|_{U_i}\colon\mathcal{O}_{U_i}\to\mathcal{L}|_{U_i}$ is an isomorphism. Accordingly, there exist unique $x_{j/i}\in\mathcal{O}_{U_i}$ such that $s_j = x_{j/i}s_i$ on U_i (we have $x_{i/i}=1$). The elements $x_{j/i}$ for a fixed i induce a morphism $f_i\colon U_i\to D(x_i)\subseteq\mathbb{P}^n$. It is then straightforward to check that the maps f_i agree on the overlaps, producing a map $f\colon X\to\mathbb{P}^n$, that $\mathcal{L}\simeq f^*\mathcal{O}(1)$, and $s_i = f^*(x_i)$. ☐

Let us reformulate what we have learned in terms of divisors. The upshot of the above lemma is that for a globally generated invertible sheaf $\mathcal{L}$ on a smooth projective variety X we get a map

$$\phi_{\mathcal{L}}\colon X\longrightarrow\mathbb{P}V^\vee, \qquad V = H^0(X,\mathcal{L}).$$

It maps a point $x\in X$ to the hyperplane in V defined as the kernel of the "evaluation at x" map $H^0(X,\mathcal{L})\to\mathcal{L}(x)\simeq k$ (which is a hyperplane precisely because $\mathcal{L}$ is globally generated).

On the other hand, since invertible sheaves correspond to linear equivalence classes of divisors (divisors modulo principal ones) we have a bijection

$$\mathbb{P}V = \{D\geq 0 : \mathcal{L}\simeq\mathcal{O}_X(D)\}$$

with the set of effective divisors in the linear equivalence class $\mathcal{L}$. This bijection identifies a nonzero $s\in H^0(X,\mathcal{L})$ (up to scaling) with the effective divisor $D=V(s)$. Thus the map $\phi_{\mathcal{L}}$ maps a point $x\in X$ to the set of those effective divisors D linearly equivalent to $\mathcal{L}$ which contain x in their support. (Compare this with the discussion at the beginning of [Hartshorne, II 6].)

19.2. Ample invertible sheaves

In algebraic geometry of the French tradition, one often turns theorems into definitions. We define an invertible sheaf to be *ample* if it satisfies the assertion of Serre's theorem (Lecture 16, Theorem 16.2.5+Remark 16.2.6).

Definition 19.2.1. Let $\mathcal{L}$ be an invertible sheaf on a projective variety X.

(a) We say that $\mathcal{L}$ is **ample** if for every coherent sheaf $\mathcal{F}$ on X there exists an n_0 such that $\mathcal{F}\otimes\mathcal{L}^n$ is globally generated for $n\geq n_0$.

(b) We say that $\mathcal{L}$ is **very ample** if there exists a closed immersion $i\colon X\hookrightarrow\mathbb{P}^n$ for some $n\geq 0$ such that $\mathcal{L}\simeq i^*\mathcal{O}(1)$.

Remark 19.2.2. Now is as good time to review our notational conventions since they might be confusing.

- The direct sum of $n \geq 1$ copies of a coherent sheaf $\mathcal{F}$ is denoted by $\mathcal{F}^n$. When we want to be more precise, we write $\mathcal{F}^{\oplus n}$ for the same thing.
- ...unless the coherent sheaf is an invertible sheaf $\mathcal{L}$, in which case $\mathcal{L}^n$ typically means the tensor power $\mathcal{L}^{\otimes n}$ (defined for all $n \in \mathbb{Z}$ with the usual convention $\mathcal{L}^{-n} = (\mathcal{L}^\vee)^n$).
- ...unless the invertible sheaf is the trivial sheaf $\mathcal{O}_X$, in which case $\mathcal{O}_X^n$ again means $\mathcal{O}_X^{\oplus n}$.
- For a divisor D and a coherent sheaf $\mathcal{F}$ we denote by $\mathcal{F}(D)$ the tensor product $\mathcal{F} \otimes \mathcal{O}_X(D)$.
- Sometimes we denote a chosen ample line bundle by $\mathcal{O}(1)$ (without referencing an ambient projective space), and then we write $\mathcal{F}(n)$ for $\mathcal{F} \otimes \mathcal{O}(1)^n$.
- We say that a divisor D is globally generated or ample if the corresponding invertible sheaf $\mathcal{O}_X(D)$ has that property.

The following lemma gathers some basic facts about ample and very ample invertible sheaves.

Lemma 19.2.3. *Let $\mathcal{L}$ be an invertible sheaf on a projective variety X.*

(a) If $\mathcal{L}$ is very ample, then it is ample.

(b) If $\mathcal{L}$ is (very) ample, then $\mathcal{L}^n$ is (very) ample for every $n \geq 1$.

(c) If $\mathcal{L}^n$ is ample for some $n \geq 1$, then $\mathcal{L}$ is ample.

(d) If $f\colon Y \to X$ is a finite morphism and $\mathcal{L}$ is ample, then $f^*\mathcal{L}$ is ample.

(e) If $\mathcal{L}$ is ample, then for every invertible sheaf $\mathcal{M}$ on X, the sheaf $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \gg 0$.

(f) If $\mathcal{L}$ is (very) ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is (very) ample.

Proof. (a) This is Serre's theorem.

(b) For "ample" this is obvious and for "very ample" it suffices to consider the case $X = \mathbb{P}^m$ and $\mathcal{L} = \mathcal{O}(1)$ (the "Veronese embedding").

(c) Suppose $\mathcal{L}^n$ is ample and let $\mathcal{F}$ be a coherent sheaf. Consider the sheaves $\mathcal{F} \otimes \mathcal{L}^i$ for $i = 0, \dots, n-1$. Then for $m \gg 0$ each $\mathcal{F} \otimes \mathcal{L}^i \otimes \mathcal{L}^{mn}$ will be globally generated, and the assertion follows by writing arbitrary integers as $mn + i$ with $i < n$.

(d) Let $\mathcal{F}$ be a coherent sheaf on Y. Then $f_*\mathcal{F}$ is a coherent sheaf on X, and hence for $n \gg 0$ the sheaf $(f_*\mathcal{F}) \otimes \mathcal{L}^n$ is globally generated. On the other hand, we have an isomorphism ("projection formula")

$$(f_*\mathcal{F}) \otimes \mathcal{L}^n \simeq f_*(\mathcal{F} \otimes (f^*\mathcal{L})^n),$$

and we want $\mathcal{F} \otimes (f^*\mathcal{L})^n$ to be globally generated. To finish we observe (renaming the sheaf) that if $f_*\mathcal{F}$ is globally generated then so is $\mathcal{F}$. Indeed, suppose we have a surjection $\mathcal{O}_X^a \to f_*\mathcal{F}$, then by adjunction we obtain a map $\mathcal{O}_Y^a = f^*(\mathcal{O}_X^a) \to \mathcal{F}$, which is again surjective.

(e) , (f) Omitted. □

Theorem 19.2.4 (Hartshorne II 7.6 + Exercise II 7.5(e))**.** *Let $\mathcal{L}$ be an invertible sheaf on a projective variety X. If $\mathcal{L}$ is ample, then $\mathcal{L}^n$ is very ample for $n \gg 0$.*

Corollary 19.2.5. *If $\mathcal{L}$ is ample, then for every coherent sheaf $\mathcal{F}$ on X there exists an n_0 such that for all $n \geq n_0$ we have $H^q(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for $q > 0$.*

Proof. If $\mathcal{L}$ is very ample, this follows from the case $X = \mathbb{P}^m$ and $\mathcal{L} = \mathcal{O}(1)$ handled previously (Lecture 17, 17.5.3). For the general case, suppose that $\mathcal{L}^m$ is very ample by the previous theorem, then apply the very ample case to the sheaves $\mathcal{F} \otimes \mathcal{L}^i$ for $i = 0, \dots, m-1$. □

19.3. Hyperplane sections and Bertini

Let $\mathcal{L}$ be a very ample invertible sheaf on X, let $s \in \Gamma(X, \mathcal{L})$ be a nonzero section, and let $Y = V(s)$ be its zero locus (considered as an effective divisor). If $i \colon X \hookrightarrow \mathbb{P}^n$ is a closed immersion for which $\mathcal{L} \simeq i^*\mathcal{O}(1)$, and if s is the image of an element (linear form) $\ell \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, then $Y = X \cap H$ for a hyperplane $H = V(\ell) \subseteq \mathbb{P}^n$. This is why the zero sets of sections of (very) ample line bundles are called *hyperplane sections*.

Theorem 19.3.1 (Bertini)**.** *Let $X \subseteq \mathbb{P}V$ (where $V \simeq k^{n+1}$) be a smooth projective variety of dimension d. Then there exists a dense open $U \subseteq \mathbb{P}V^\vee$ such that for every hyperplane $H \in U$ the intersection $H \cap X$ is smooth of dimension $d-1$.*

Proof. The key point is that $H \cap X$ is a smooth divisor in a neighborhood of a point $x \in H \cap X$ if and only if H does not contain the tangent space T_xX. Consider the set of "bad pairs"

$$Z = \{(x, H) \in X \times \mathbb{P}V^\vee : T_xX \subseteq H\},$$

a closed subset of $X \times \mathbb{P}V^\vee$. Then the assertion holds for $H \in \mathbb{P}V^\vee$ if and only if $H \notin q(Z)$ where $q \colon Z \to \mathbb{P}V^\vee$ is the projection $(x, H) \mapsto H$. The subset $q(Z) \subseteq \mathbb{P}V^\vee$ is closed (since X is complete, so that q is a closed map), and the desired open U will be its complement. Thus, we must show that $q(Z)$ is not the whole of $\mathbb{P}V^\vee$.

We will prove that $q(Z) \neq \mathbb{P}V^\vee$ by showing that $\dim(Z) < n = \dim \mathbb{P}V^\vee$. To this end, consider the other projection $p \colon Z \to X$ given by $(x, H) \mapsto x$. The fiber of p above $x \in X$ is the set of hyperplanes in $\mathbb{P}V \simeq \mathbb{P}^n$ containing the d-dimensional linear subspace T_xX. It can be identified with the set of hyperplanes in the quotient space $V/C(T_xX)$ where $C(T_xX)$ is the cone over T_xX, a linear subspace of V of dimension $d+1$. Thus the fiber $p^{-1}(x)$ can be identified with $\mathbb{P}(V/C(T_xX))^\vee \simeq \mathbb{P}^{n-d-1}$, which has dimension $n-d-1$. Consequently, we have

$$\dim(Z) = \dim(X) + \dim(\text{general fiber of } p) = d + (n-d-1) = n-1 < n. \qquad \square$$

Corollary 19.3.2. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety X. Then there exist smooth prime divisors $D_1, \dots, D_r, E_1, \dots, E_s$ on X such that $D_i \cap D_j = \emptyset = E_i \cap E_j$ for $i \neq j$ such that*

$$\mathcal{L} \simeq \mathcal{O}_X((D_1 + \cdots + D_r) - (E_1 + \cdots + E_s)).$$

Remark 19.3.3. We can choose the $D_1, \dots, D_r$ and $E_1, \dots, E_s$ to be ample and if $\dim(X) \geq 2$ we will have $r = s = 1$.

Proof. Let $\mathcal{M}$ be a very ample invertible sheaf. Choose $n > 0$ such that both $\mathcal{M}^n$ and $\mathcal{M}^n \otimes \mathcal{L}$ are very ample (cf. Lemma 19.2.3(e)). By Bertini (Theorem 19.3.1) we can find $s \in H^0(X, \mathcal{M}^n \otimes \mathcal{L})$ and $t \in H^0(X, \mathcal{M}^n)$ such that $D = V(s)$ and $E = V(t)$ are smooth. Then $\mathcal{M}^n \otimes \mathcal{L} \simeq \mathcal{O}_X(D)$ and $\mathcal{M}^n \simeq \mathcal{O}_X(E)$, so that $\mathcal{L} \simeq \mathcal{O}_X(D - E)$. □

19.4. Intersections on surfaces

Let X be a smooth projective surface and let C and D be two divisors on X. Our goal is to associate to this pair an integer $(C.D)$ called the **intersection number**. Our basic requirement is that if C and D are distinct prime divisors (so that $C\cap D$ is a finite set), then $(C.D)$ is the number of points in the intersection, counted with appropriate multiplicity:

$$(C.D)=\sum_{x\in C\cap D} i(C,D;x), \qquad i(C,D;x)=\dim_k \mathcal{O}_{X,x}/(\mathcal{I}_{C,x}+\mathcal{I}_{D,x}).$$

We also want $(C.D)$ to be bilinear and symmetric, and to depend only on the linear equivalence classes of C and D.

Suppose that C and D are smooth prime divisors intersecting transversally (meaning that their tangent lines $T_xC, T_xD\subseteq T_xX$ are distinct), meaning that $i(C,D;x)=1$ for every $x\in C\cap D$. (We make this assumption in order for $C\cap D$ to be reduced, so that we can avoid talking about schemes.) The trick to extending $(C.D)$ to arbitrary divisors is to regard $\#(C\cap D)$ as the Euler characteristic $\chi(\mathcal{O}_{C\cap D})$ (the higher cohomology groups of $\mathcal{O}_{C\cap D}$ are zero). In this case we have an exact sequence (a special case of a "Koszul complex")[1]

$$0\longrightarrow \mathcal{O}_X(-C-D)\longrightarrow \mathcal{O}_X(-C)\oplus\mathcal{O}_X(-D)\longrightarrow \mathcal{O}_X\longrightarrow \mathcal{O}_{C\cap D}\longrightarrow 0$$

(recall that $\mathcal{I}_C=\mathcal{O}_X(-C)$ and $\mathcal{I}_D=\mathcal{O}_X(-D)$), from which we infer that

$$(C.D)=\chi(\mathcal{O}_{C\cap D})=\chi(\mathcal{O}_X)-\chi(\mathcal{O}_X(-C))-\chi(\mathcal{O}_X(-D))+\chi(\mathcal{O}_X(-C-D)).$$

The same argument (now with multiplicities taken into account) applies to an arbitrary pair of curves C and D without common components (this ensures the exactness of the Koszul complex on the left). This motivates the following definition (cf. Kempf §10.9, Hartshorne V Ex. 1.1):

Definition 19.4.1. Let C and D be divisors on a smooth projective surface X. Their **intersection number** is defined as

$$(C.D)=\chi(\mathcal{O}_X)-\chi(\mathcal{O}_X(-C))-\chi(\mathcal{O}_X(-D))+\chi(\mathcal{O}_X(-C-D)).$$

Clearly, the intersection number $(C.D)$ depends only on the linear equivalence classes of C and D and is symmetric. Its being bilinear follows from the lemma below.

Lemma 19.4.2. *Suppose that $D=D_1+\cdots+D_r-E_1-\cdots-E_s$ for smooth curves $D_1,\dots,D_r,E_1,\dots,E_s$ as in Corollary 19.3.2. Then for every divisor C we have*

$$(C.D)=\sum\deg(\mathcal{O}_X(C)|_{D_i})-\sum\deg(\mathcal{O}_X(C)|_{E_i}).$$

In particular, if D is a smooth curve, we have $(C.D)=\deg(\mathcal{O}_X(C)|_D)$.

Proof. The proof follows a straightforward calculation using various restriction exact sequences — see [Kempf, Lemma 10.9.2]. □

Example 19.4.3 (Bézout theorem)**.** On $X=\mathbb{P}^2$, we have an isomorphism $\deg\colon \mathrm{Pic}(X)\xrightarrow{\sim}\mathbb{Z}$ sending $\mathcal{O}(1)$ to 1. Then $(C.D)=\deg(C)\cdot\deg(D)$. This implies that two irreducible curves $C,D\subseteq X$ without common components have exactly $\deg(C)\cdot\deg(D)$ points in common, counted with multiplicity.

[1] See the last subsection.

Example 19.4.4 (Exceptional divisor)**.** Let $\pi\colon X = \mathrm{Bl}_P\mathbb{P}^2 \to \mathbb{P}^2$ be the blowup of a point P in $\mathbb{P}^2$. The fiber $E = \pi^{-1}(P)$ is isomorphic to $\mathbb{P}^1$. We shall prove that $(E.E) = -1$. Since this is negative, we conclude from this that there is no effective $E' \sim E$ with $E' \neq E$ (otherwise $(E.E) = (E.E')$ would count the points in $E \cap E'$ and hence be non-negative).

Let $L \subseteq \mathbb{P}^2$ be a line through P and let $L' \subseteq \mathbb{P}^2$ be a map not through P. Let $\tilde{L} \subseteq X$ and $\tilde{L}' \subseteq X$ be their strict transforms. Then $\pi^{-1}(L) = \tilde{L} + E$ and $\pi^{-1}(L') = \tilde{L}'$. Since $\mathcal{O}_{\mathbb{P}^2}(L) = \mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^2}(L')$, we have

$$\mathcal{O}_X(\tilde{L}+E) \simeq \pi^*\mathcal{O}_{\mathbb{P}^2}(L) \simeq \pi^*\mathcal{O}_{\mathbb{P}^2}(L') \simeq \mathcal{O}_X(\tilde{L}'),$$

i.e. $\tilde{L}+E \sim \tilde{L}'$. Thus

$$\underbrace{(\tilde{L}'.\tilde{L}')}_{=1} = \underbrace{(\tilde{L}.\tilde{L})}_{=0} + 2\underbrace{(\tilde{L}.E)}_{=1} + (E.E),$$

and hence $(E.E) = -1$.

Example 19.4.5 (The diagonal)**.** Let C be a curve of genus g, let $X = C \times C$, and let $\Delta\colon C \hookrightarrow X$ be the diagonal. Then $\mathcal{O}(-\Delta) \simeq \omega_C$. It follows that

$$(\Delta.\Delta) = \deg(\mathcal{O}_X(\Delta)|_\Delta) = \deg(\omega_C^{-1}) = 2-2g.$$

Example 19.4.6. Let $f\colon Y \to X$ be a finite flat morphism of surfaces of degree d and let C and D be divisors on X. Then

$$(f^*C.f^*D) = d \cdot (C.D).$$

Lemma 19.4.7. *If $\mathcal{L} = \mathcal{O}_X(C)$ is ample and D is effective then $(C.D) \geq 0$, with equality if and only if $D = 0$. Moreover, we have $(C.C) > 0$.*

Proof. If C is ample then for every smooth curve $D \subseteq X$ the restriction $\mathcal{L}|_D$ is ample, and hence $(C.D) = \deg(\mathcal{L}|_D) > 0$. For the general case, replacing C with nC for $n \gg 0$ we may assume that C is very ample, in which case we can find a smooth curve $C' \sim C$ which intersects D transversally. See [Hartshorne, V Lemma 1.3] for the details. □

The following theorem gives a converse to the above lemma.

Theorem 19.4.8 (Nakai–Moishezon criterion for ampleness, Hartshorne V Theorem 1.10)**.** *For a divisor C on X, the invertible sheaf $\mathcal{O}_X(C)$ is ample if and only if $(D.D) > 0$ and $(C.D) > 0$ for every irreducible curve $C \subseteq X$.*

The following theorem will be proved next week. It states that the intersection form on $\mathrm{Pic}(X)$ has "signature $(1, n-1)$." We cannot formulate it like that since $\mathrm{Pic}(X)$ is not a finite-dimensional vector space (though it can be done, see next week). By Lemma 19.4.7 above, we have $(H.H) > 0$ if $\mathcal{O}_X(H)$ is ample, thus having "signature $(1, n-1)$" should mean that the quadratic form $(D.D)$ is negative-definite on the orthogonal to H:

Theorem 19.4.9 (Hodge index theorem)**.** *If H is a divisor such that $\mathcal{O}_X(D)$ is ample and D a divisor such that $(D.H) = 0$, then $(D.D) \leq 0$, with equality if and only if $(D.C) = 0$ for every divisor C.*

19.5. Problem session (Apr 14)

During the problem session we proved assertions (e) and (f) in Lemma 19.2.3. We then went through the material in §19.4 (the lectured covered only the first three subsections), proving the exactness of the Koszul complex and deducing the formula for $(C.D)$ in terms of χ. We then proved Lemma 19.4.2 and deduced the formula in Example 19.4.5.

19.6. Bonus: exactness of the Koszul complex

The following result was used implicitly at the beginning of §19.4.

Definition 19.6.1. Let A be a local ring and let $x_1,\dots,x_n \in \mathfrak{m}_A$ be elements in its maximal ideal. We say that $(x_1,\dots,x_n)$ form a **regular sequence** if for every $i = 1,\dots,n$, the image of x_i in $A/(x_1,\dots,x_{i-1})$ is a nonzerodivisor. If A is Noetherian, we say that A satisfies **Serre's condition** (S_d) if it admits a regular sequence of length $n = \inf\{\dim(A), d\}$. We say that A is **Cohen–Macaulay** if it is Noetherian and admits a regular sequence of length $n = \dim(A)$.

Let A be any ring and let $x_1,\dots,x_n \in A$. For $0 \le m \le n$, let $\bigwedge^m A^n$ denote the free A-module of rank $\binom{n}{m}$ with basis $e_I = e_{i_1} \wedge \dots \wedge e_{i_m}$ where $I = \{i_1,\dots,i_n\}$, $1 \le i_1 < \dots < i_m \le n$. We define the differential $d\colon \bigwedge^m A^n \to \bigwedge^{m-1} A^n$ by the formula

$$d(e_I) = \sum_{i\in I} (-1)^j x_i \cdot e_{I\setminus\{i\}}.$$

For example, for $m = 1$, $d\colon A^n \to A$ is the map $(x_1,\dots,x_n)$. The **Koszul complex** is the resulting complex

$$K^\bullet(A;x_1,\dots,x_n) = \Big[\ \bigwedge^n A^n \longrightarrow \bigwedge^{n-1} A^n \longrightarrow \dots \longrightarrow \bigwedge^2 A^n \longrightarrow A^n \longrightarrow A\ \Big],$$

placed in degrees $[-n,0]$. The Koszul complex is independent of the order of $x_1,\dots,x_n$, up to changing signs in the differentials. The projection $A \to A/(x_1,\dots,x_n)$ gives a map $K^\bullet(A;x_1,\dots,x_n) \to A/(x_1,\dots,x_n)$ (the target placed in degree zero).

Proposition 19.6.2. *Let A be a Noetherian local ring and $x_1,\dots,x_n \in \mathfrak{m}_A$. The following are equivalent:*

- *(a) $(x_1,\dots,x_n)$ is a regular sequence;*
- *(b) $K^\bullet(A;x_1,\dots,x_n) \to A/(x_1,\dots,x_n)$ is a quasi-isomorphism (equivalently, $H^i(K^\bullet(A;x_1,\dots,x_n)) = 0$ for $i \neq 0$.*

Proof. For $n = 1$, the Koszul complex is

$$K^\bullet(A,x_1) = [\ A \xrightarrow{x_1} A\ ],$$

and the equivalence of (a) and (b) is clear. For the general case, note first that $(x_1,\dots,x_n)$ is a regular sequence if and only if x_1 is a nonzerodivisor and $(x_2,\dots,x_n)$ gives a regular sequence on A/x_1. We shall complete the argument for (a)⇒(b) for $n = 2$, which is sufficient for §19.4 (see [Eisenbud, §17] for the general case). Tensoring the short exact sequences

$$0 \longrightarrow A \xrightarrow{x_i} A \longrightarrow A/x_i \longrightarrow 0$$

together, we obtain a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & A & \xrightarrow{x_1} & A & \longrightarrow & A/x_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_2\,\mid\,\alpha} & & \\
0 & \longrightarrow & A & \xrightarrow{x_1} & A & \longrightarrow & A/x_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \cdots\to & A/x_2 & \xrightarrow[\beta]{x_1} & A/x_2 & \longrightarrow & A/(x_1,x_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Snake lemma gives $\ker(\alpha) \xrightarrow{\sim} \ker(\beta)$, i.e. x_1 is a nonzerodivisor in A/x_2 if and only if x_2 is a nonzerodivisor in A/x_1. Thus if (x_1, x_2) is a regular sequence, then the diagram is exact also with the dotted arrows. A diagram chase then shows that

$$K^\bullet(A; x_1, x_2) = \left[A \xrightarrow{(x_1, -x_2)^T} A^2 \xrightarrow{(x_1, x_2)} A \right]$$

is exact in negative degrees, so that (b) holds. □