2. Lecture 2 (Jan 15)

Recommended reading: Kempf §1, Hartshorne I.3

2.1. Some general topology

Definition 2.1.1. Let X be a topological space.

(a) We say that X is **irreducible** if for every pair of closed subsets $Y_1, Y_2 \subseteq X$ with $X = Y_1 \cup Y_2$, we have $X = Y_1$ or $X = Y_2$. (Equivalently, every non-empty open subset of X is dense.)

(b) The **dimension** $\dim(X)$ is the supremum of the set of integers $n \geq 0$ for which there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of distinct closed irreducible subsets of X.

(c) We say that X is **Noetherian** if every decreasing sequence $F_0 \supseteq F_1 \supseteq \ldots$ of closed subsets stabilizes (is eventually constant).

Proposition 2.1.2. *Let $X = V(f_1, \ldots, f_r) \subseteq k^n$ be an affine algebraic set and let $A = k[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$. Then:*

(a) X is irreducible if and only if the coordinate ring

$$\mathcal{O}(X) = k[T_1, \ldots, T_n]/\mathcal{I}(X) = k[T_1, \ldots, T_n]/\sqrt{(f_1, \ldots, f_r)} = A/\sqrt{0}$$

is a domain, or equivalently iff the ideal $\sqrt{(f_1, \ldots, f_r)} = \mathcal{I}(X) \subseteq k[T_1, \ldots, T_n]$ is prime.

(b) X is a Noetherian topological space.

(c) We have $\dim(X) = \dim(\mathcal{O}(X)) = \dim(A)$ *(the latter two denote the* **Krull dimension***).*

Remark 2.1.3. Thus irreducible closed subsets of X correspond to prime ideals of $\mathrm{Spec}(A)$. The smallest irreducible closed subsets are the points of X, which correspond to the largest prime ideals, i.e. the maximal ideals.

Lemma 2.1.4. *Let X be a Noetherian topological space. Then there exist closed irreducible subsets $Z_1, \ldots, Z_r \subseteq X$ such that $Z_i \not\subseteq Z_j$ for $i \neq j$ and*

$$X = Z_1 \cup \cdots \cup Z_r.$$

They are unique up to permutation.

Definition 2.1.5. The closed irreducible subsets $Z_1, \ldots, Z_r \subseteq X$ in the lemma are called the **irreducible components** of X.

Note that we have $\dim(X) = \sup\{\dim Z_1, \ldots, \dim Z_r\}$. In order to say more about dimension, we need to review some results from the **dimension theory** part of commutative algebra (see e.g. Chapter 10 of Atiyah–Macdonald, though we need a bit more).

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set, let $\mathfrak{p} = \mathcal{I}(X) \subseteq k[T_1, \ldots, T_n]$ be the corresponding prime ideal, and let $A = k[T_1, \ldots, T_n]/\mathfrak{p} = \mathcal{O}(X)$.

(1) Let $K = \mathrm{Frac}(A)$ be the field of fractions. Then

$$\dim(X) = \dim(A) = \mathrm{trdeg}(K/k)$$

is the transcendence degree of the extension K/k. In particular, we have $\dim \mathbb{A}^n = n$.

(2) All maximal chains $Z_0 \subsetneq \cdots \subsetneq Z_r$ of irreducible subsets of X have length $r = \dim(X)$.

From this we can deduce that for a locally closed $Y \subseteq \mathbb{A}^n$, we have $\dim(Y) = \dim(\overline{Y})$.

(3) Let $f \in A$ be a nonzero nonunit, and let $Y = V(f) \subseteq X$. Let $Y_1, \ldots, Y_r \subseteq Y$ be the irreducible components of Y. Then

(a) We have $\dim(Y_i) = \dim(X) - 1$ for every $i = 1, \ldots, r$.

(b) Conversely, A is a UFD (unique factorization domain, for example if $X = \mathbb{A}^n$), then every closed irreducible subset $Y \subseteq X$ such that $\dim(Y) = \dim(X) - 1$ is of the form $Y = V(f)$ for some prime element $f \in A$.

2.2. Regular functions

A polynomial $f \in k[T_1, \ldots, T_n]$ defines a function $f\colon k^n \to k$. Similarly, a rational function $f = p/q \in k(T_1, \ldots, T_n)$ with $p, q \in k[T_1, \ldots, T_n]$, $q \neq 0$ defines a function $f = p/q\colon D(q) \to k$ where $D(q) = k^n \setminus V(q)$. Recall that the subsets $D(q)$ for a varying q form a basis of the Zariski topology on $k^n = \mathbb{A}^n$. We define regular functions as those which are *locally* given by a rational function.

Definition 2.2.1. Let $X \subseteq k^n$ be an algebraic set, $U \subseteq X$ an open subset, and let $f\colon U \to k$ be a function. We say that f is a **regular function** on U if every $x \in U$ there exists an open neighborhood $x \in V \subseteq U$ and $p, q \in k[T_1, \ldots, T_n]$ such that for every $y \in V$, we have $q(y) \neq 0$ and

$$f(y) = \frac{p(y)}{q(y)}.$$

We observe first that if f is a regular function on U and $Z \subseteq U$ is a locally closed subset, then $f|_Z$ is a regular function. Moreover, every polynomial $f \in k[T_1, \ldots, T_n]$ defines a regular function on $\mathbb{A}^n$ and hence on every locally closed $U \subseteq \mathbb{A}^n$. For an affine algebraic set $X \subseteq \mathbb{A}^n$ we obtain a map

$$A = \mathcal{O}(X) = k[T_1, \ldots, T_n]/\mathcal{I}(X) \longrightarrow \{\text{regular functions on } X\}.$$

It follows from the Nullstellensatz that this map is injective. Moreover, regular functions on any locally closed subset $U \subseteq \mathbb{A}^n$ form a k-subalgebra of the ring of all functions $U \to k$.

Theorem 2.2.2. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set and let $A = k[T_1, \ldots, T_n]/\mathcal{I}(X)$ be its coordinate ring. Then:*

(a) The map

$$A \longrightarrow \{\text{regular functions on } X\}$$

is an isomorphism of k-algebras.

(b) For $g \in A$, let $U = D(g) \subseteq X$. Then the above map induces an isomorphism

$$A[g^{-1}] \simeq \{\text{regular functions on } U\}.$$

The proof shows an algebraic variant of "partitions of unity" in differential geometry.

Proof. (a) We need to show that this map is surjective, so let f be a regular function on X and let $X = \bigcup V_i$ be an open cover such that $f|V_i = p_i/q_i$ for $p_i, q_i \in k[T_1, \dots, T_n]$ with $V_i \subseteq D(q_i)$. We may assume that $V_i = D(g_i)$ for some $g_1, \dots, g_r \in A$ generating the unit ideal in A. We can simplify this a bit further: replacing g_i with $g_i q_i$ and p_i/q_i with $(p_i g_i)/(q_i g_i)$ we may assume that $g_i = q_i$. Consider the functions

$$f_i = q_i^2 f \colon X \to k.$$

We notice that $f_i = p_i q_i$ for every i (the right-hand side is the function $X \to k$ defined by the element $p_i q_i \in k[T_1, \dots, T_n]$). Indeed, on V_i we have $f_i = q_i^2(p_i/q_i) = p_i q_i$, and outside of V_i both sides are zero.

Since $A = (q_1, \dots, q_r)$, we also have $A = (q_1^2, \dots, q_r^2)$. Let $a_1, \dots, a_r \in A$ be such that $1 = \sum a_i q_i^2$. Multiply the last equality by f to get

$$f = \sum_{i=1}^{r} a_i q_i^2 f = \sum_{i=1}^{r} a_i f_i = \sum_{i=1}^{r} a_i p_i q_i \in A.$$

(b) We apply (a) to $V(I, T_{n+1} g - 1) \subseteq \mathbb{A}^{n+1}$. □

2.3. Spaces with functions

Definition 2.3.1. Fix a field k. A **space with functions** (**swf** for short) is a topological space X together with an assignement, for every open $U \subseteq X$, of a k-subalgebra $\mathcal{O}(U)$ of the ring of all functions $U \to k$ (called the ring of **regular functions** on U) such that

(a) Being regular is a local property. That is, if $U = \bigcup U_\alpha$ is an open cover and $f \colon U \to k$ is a function, then $f \in \mathcal{O}(U)$ if and only if $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for each α.

(b) If $U \subseteq X$ is an open subset and $f \in \mathcal{O}(U)$, then the set

$$D(f) = \{x \in U : f(x) \neq 0\} \subseteq U$$

is an open subset of U and $(f|_{D(f)})^{-1} \in \mathcal{O}(D(f))$.

A **morphism** of spaces with functions is a continuous map $\phi \colon Y \to X$ such that the pullbacks of regular funtions are regular: for every open $U \subseteq X$ and every regular function $f \in \mathcal{O}(U)$, the function $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$.

Note that a map of swf's $\phi \colon Y \to X$ induces a k-algebra homomorphism

$$\phi^* \colon \mathcal{O}(X) \longrightarrow \mathcal{O}(Y), \qquad \phi^*(f) = f \circ \phi.$$

Remark 2.3.2 (If you know some sheaf theory)**.** Condition (a) means that $\mathcal{O}$ forms a *subsheaf* of the sheaf $\prod_{x \in X} k_x$ of all k-valued functions on X. Condition (b) ensures that the stalks $\mathcal{O}_x = \varinjlim_{x \in U} \mathcal{O}(U)$ for $x \in X$ are local rings, with maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_x : f(x) = 0\}$.

Examples 2.3.3. (a) Let $k = \mathbb{R}$ or $\mathbb{C}$ and let X be a topological space. Then $\mathcal{O}(U) = C(U; k)$ (continuous functions $U \to k$) gives X the structure of a space with functions.

(b) Similarly with C^∞, analytic, and complex manifolds.

(c) If X is a space with functions and $U \subseteq X$ is an open subset, then U is a space with functions in the obvious way.

(d) Let again k be our chosen algebraically closed field and let $X \subseteq \mathbb{A}^n$ be a locally closed subset. For an open $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions on U as in Definition 2.3.1. This makes X into a space with functions. Note that by Theorem 2.2.2, the two meanings of $\mathcal{O}(X)$ we have introduced agree.

Theorem 2.3.4. *Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then for every space with functions Y, the pull-back map*

$$\phi \mapsto \phi^* \colon \operatorname{Hom}(Y,X) \longrightarrow \operatorname{Hom}_k(\mathcal{O}(X),\mathcal{O}(Y))$$

is bijective.

Note that by Yoneda's lemma, this determines the swf X if we know the ring $\mathcal{O}(X)$. (There may be other swfs with the same $\mathcal{O}(X)$, but only one of them is an affine algebraic set.)

Proof. **Injectivity** is easy: if $\phi, \psi \colon Y \to X$ are two maps and $y \in Y$ is such that $\phi(y) = x \neq x' = \psi(y)$, we find an $f \in \mathcal{O}(X)$ with $f(x) \neq f(x')$ (for example, one of the coordinates $T_1, \dots, T_n$), and then $\phi^*(f)(y) = f(x) \neq f(x') = \psi^*(f)(y)$, and $\phi^* \neq \psi^*$.

Surjectivity: Let $\phi^* \colon \mathcal{O}(X) \to \mathcal{O}(Y)$ be a k-algebra homomorphism, for which we seek to build the corresponding map of swf's $\phi \colon Y \to X$. For each $y \in Y$, we have the evaluation map $\mathrm{ev}_y \colon \mathcal{O}(Y) \to k$ mapping $f \mapsto f(y)$. Consider the composition

$$\mathcal{O}(X) \xrightarrow{\phi^*} \mathcal{O}(Y) \xrightarrow{\mathrm{ev}_y} k.$$

This defines an element $x \in \operatorname{Hom}_k(\mathcal{O}(X),k)$, which equals X by the Nullstellensatz. We define the map ϕ by $\phi(y) = x$. This defines a map (of sets) $\phi \colon Y \to X$ inducing ϕ^*. Moreover, the pull-back of the basic open set $D(f) \subseteq X$ is $D(\phi^* f) \subseteq Y$, which is open by axiom (b) of the definition of an swf, which shows that ϕ is continuous. We omit the (easy) verification that ϕ is a morphism of swf's. □

Remark 2.3.5. Here is a direct way of reconstructing the swf X from the reduced k-algebra $A = \mathcal{O}(X)$. We set $X = \operatorname{MSpec}(A) = \operatorname{Hom}_k(A,k)$. We give it the induced topology from $\operatorname{Spec}(A)$, in other words generated by the base open sets $D(g)$ for $g \in A$. Finally, we call a function $f \colon U \to k$ defined on an open $U \subseteq X$ regular for every $x \in U$ there exist $g, h \in A$ such that $D(g) \subseteq U$ and $f(y) = h(y)/g(y)$ for every $y \in D(g)$.

Corollary 2.3.6. *The category of* **affine algebraic sets** *(defined as a full subcategory of the category of swf's) is equivalent to the opposite category of the category of* **finitely generated reduced k-algebras**.

Corollary 2.3.7 (Products of affine algebraic sets)**.** *The category of affine algebraic sets admits products. More precisely, let $X = V(I) \subseteq \mathbb{A}^n$ (with coordinates $T_1, \dots, T_n$) and $Y = V(J) \subseteq \mathbb{A}^m$ (with coordinates $U_1, \dots, U_m$) be two affine algebraic sets. Then*

$$X \times Y = V(I+J) \subseteq \mathbb{A}^{n+m}$$

is the product of X and Y in the category of swf's, and we have

$$\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

Proof. This is straightforward except for the fact that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. For this, see Proposition 5.17 in Milne's notes[1] (which also shows that $\mathcal{O}(X \times Y)$ is a domain if $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains. □

[1] https://www.jmilne.org/math/CourseNotes/AG.pdf

We are finally able to define algebraic sets (which we will later identify as reduced schemes of finite type over k) and varieties.

Definition 2.3.8. Let k be an algebraically closed field.

(1) An **algebraic set** over k is a space with functions X admitting a finite open cover $X = U_1 \cup \cdots \cup U_n$ where each U_i is isomorphic as an swf to an affine algebraic set.

(2) We say that an algebraic set is a **variety** if it is irreducible.

(3) We say that X is **projective** if it is isomorphic to a projective algebraic set (see below).

(4) We say that X is **quasi-affine** if it is isomorphic to an open subset of an affine algebraic set, and **quasi-projective** if it is isomorphic to an open subset of a projective algebraic set.

We note the key fact that not only can every algebraic set be covered by open affine algebraic sets, but the **affine open subsets form a base** for the topology on X (since if $U \subseteq X$ is affine, then a basis of opens of U is given by the sets $D(g)$ for $g \in \mathcal{O}(U)$).

Example 2.3.9. The projective space $\mathbb{P}^n$ has the standard open cover $U_0 \cup \cdots \cup U_n$ by affine spaces. We call a function on a locally closed subset Z of $\mathbb{P}^n$ regular if its restriction to each $Z \cap U_i$ is regular in the sense of Definition 2.3.1. Thus every locally closed subset of $\mathbb{P}^n$ is an swf and moreover an algebraic set.

Example 2.3.10. The punctured plane $\mathbb{A}^2 \setminus \{0\}$ is quasi-affine but not affine (see below). Similarly, the punctured projective plane $\mathbb{P}^2 \setminus \{P\}$ for a point P is quasi-projective, but neither projective nor quasi-affine.

Lemma 2.3.11. *(a)* $\mathcal{O}(\mathbb{A}^{n+1} \setminus 0) = \mathcal{O}(\mathbb{A}^{n+1})$ *for* $n \geq 1$*;*

(b) $\mathcal{O}(\mathbb{P}^n) = k$*;*

(c) Let $X \subseteq \mathbb{P}^n$ *be a projective variety (closed and irreducible subset). Then* $\mathcal{O}(X) = k$*.*

Proof. (a) The set $U = \mathbb{A}^{n+1} \setminus 0$ is the union of $D(T_i)$, $i = 0, \ldots, n$. We have $\mathcal{O}(D(T_i)) = \mathcal{O}(\mathbb{A}^{n+1})[T_i^{-1}]$. Consider all of these as subrings of $k[T_0^{\pm 1}, T_n^{\pm 1}]$, then $\mathcal{O}(U)$ is their intersection, which equals $k[T_0, \ldots, T_n] = \mathcal{O}(\mathbb{A}^{n+1})$.

(b) For this we use the fact (easy proof omitted) that for an open (or locally closed) $W \subseteq \mathbb{P}^n$, a function $f\colon W \to k$ is regular if and only if $f \circ \pi$ is regular on $\pi^{-1}(W)$ where $\pi\colon U \to \mathbb{P}^n$ is the quotient map. The fact for $W = \mathbb{P}^n$ combined with (a) implies that $\mathcal{O}(\mathbb{P}^n)$ consists of all $f \in k[T_0, \ldots, T_n]$ which are invariant under scaling of the coordinates, i.e. homogeneous of degree zero. But $k[T_0, \ldots, T_n]_0 = k$.

(c) We shall prove this later. □

Example 2.3.12 (Ojanguren)**.** In all examples of algebraic sets we have encountered so far, the ring $\mathcal{O}(X)$ was a finitely generated k-algebra. This is true for affine algebraic sets and projective algebraic sets, but for completely different reasons. In general for an algebraic set X, the ring $\mathcal{O}(X)$ might be non-Noetherian. Here is a simple example, found by Ojanguren. Consider the projective three-space $\mathbb{P}^3$ with homogeneous coordinates $(X : Y : Z : T)$ and the subsets

$$W = V_{\mathbb{P}}(XY) \subseteq \mathbb{P}^3, \qquad L = V_{\mathbb{P}}(X, Z) \subseteq W, \qquad U = X \setminus L.$$

Thus W is the union of two hyperplanes $H_1 = V_{\mathbb{P}}(X), H_2 = V_{\mathbb{P}}(Y) \simeq \mathbb{P}^2$ in $\mathbb{P}^3$ intesecting along the line $V_{\mathbb{P}}(X, Y) \simeq \mathbb{P}^1$. The set $L \simeq \mathbb{P}^1$ is another line, contained in one of the planes H_1 and intersecting the other H_2 in a single point $Q = (0 : 0 : 0 : 1)$. Let us calculate $\mathcal{O}(U)$. A regular function f on U restricts to a regular function f_1 on $H_1 \setminus L \simeq \mathbb{A}^2$ (with coordinates $u = Y/Z$ and $v = T/Z$) and a regular function f_2 on $H_2 \setminus Z \simeq \mathbb{P}^2 \setminus Q$. But $\mathcal{O}(\mathbb{P}^2 \setminus Q) = k$, so f_2 is constant. It follows that

$$\mathcal{O}(U) = \{f \in k[u,v] \,:\, f(0,v) \in k\}$$

which is not Noetherian (as the ideal $v \cdot k[u,v]$ is contained in $\mathcal{O}(U)$ and is an ideal there, generated by vu^n for all $n \geq 0$ but not by any proper subset).