20. Lecture 20 (Apr 16): Surfaces

In the final lecture of the "core" part of the course, we further develop intersection theory on surfaces: Serre duality, Riemann–Roch, and the Hodge index theorem.

Recommended reading: Hartshorne V 1, Kempf §10

20.1. General Serre duality

Theorem 20.1.1. *Let X be a smooth projective variety of dimension d. For every locally free sheaf $\mathcal{E}$ on X and every $0 \leq q \leq d$ there exists a canonical isomorphism*

$$H^q(X, \mathcal{E}) \simeq H^{n-q}(X, \mathcal{E}^\vee \otimes \omega_X)^\vee.$$

Remark 20.1.2. We shall only need the following corollary: if $\mathcal{L}$ is an invertible sheaf on a smooth projective surface X, then

$$\dim H^2(X, \mathcal{L}) = \dim H^0(X, \mathcal{L}^{-1} \otimes \omega_X).$$

I do not know if there is a simple proof of this.

The following application of coherent cohomology was promised earlier in Lecture 15, Example 15.1.1.

Corollary 20.1.3. *Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety of dimension ≥ 2 and let $H \subseteq \mathbb{P}^n$ be a hyperplane. Then the intersection $H \cap X$ is connected.*

Proof. As explained in Lecture 15, Example 15.1.1, the result will follow once we know that $H^1(X, \mathcal{O}_X(-n)) = 0$ for $n \gg 0$. By Serre duality, this space is dual to $H^{\dim(X)-1}(X, \mathcal{O}_X(n) \otimes \omega_X)$. Since $\dim(X) - 1 > 0$, this last group will vanish for $n \gg 0$ (see Lecture 19, Lemma 19.2.2(d)). □

20.2. Riemann–Roch for surfaces

Let X be a smooth projective surface. As in the case of curves, we fix a divisor K for which $\mathcal{O}_X(K) \simeq \omega_X$ and call it a/the canonical divisor.

Last time, we introduced the intersection numbers $(C.D)$ between divisors on X. Let us recall their properties in form of a theorem.

Theorem 20.2.1. *Let X be a smooth projective surface. There exists a unique symmetric bilinear pairing*

$$(-.-)\colon \mathrm{Div}(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}$$

satisfying the following two properties:

i. if $C \sim C'$ and $D \sim D'$ (linear equivalence), then $(C.D) = (C'.D')$;

ii. if C and D are distinct prime divisors (irreducible curves), then

$$(C.D) = \dim_k H^0(X, \mathcal{O}_{C\cap D}) = \sum_{x \in C\cap D} i(C,D;x), \qquad i(C,D;x) = \dim_k \mathcal{O}_{C\cap D,x},$$

where $\mathcal{O}_{C\cap D} = \mathcal{O}_X/\mathcal{I}_{C\cap D}$ for $\mathcal{I}_{C\cap D} = \mathcal{I}_C + \mathcal{I}_D$ (since we are not taking the radical, we are counting the intersection points with the correct multiplicity).

In addition, the intersection numbers $(C.D)$ enjoy the following properties:

(a) For any divisors C and D, we have

$$(C.D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C-D)).$$

(b) If C is a smooth prime divisor, then for every divisor D we have

$$(C.D) = \deg(\mathcal{O}_X(D)|_C).$$

The Riemann–Roch theorem has a version for surfaces.

Theorem 20.2.2 (Riemann–Roch for surfaces)**.** *Let X be a smooth projective surface and let D be a divisor on X. Then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D.(D-K)) + \chi(\mathcal{O}_X).$$

Proof. Let us try to follow the strategy of proof of the Riemann–Roch theorem for curves. The statement holds trivially for $D = 0$, and it depends only on the linear equivalence class of D. Applying Bertini (Lecture 19, Corollary 19.3.2) inductively, we see that it suffices to show that if $C \subseteq X$ is a smooth curve, then the statement holds for D if and only if it holds for $D+C$, or equivalently that

$$\chi(\mathcal{O}_X(D+C)) - \chi(\mathcal{O}_X(D)) = \left(\frac{1}{2}((D+C).(D+C-K)) + \chi(\mathcal{O}_X)\right) - \left(\frac{1}{2}(D.(D-K)) + \chi(\mathcal{O}_X)\right),$$

where the right-hand side simplifies further as $(C.D) - \frac{1}{2}(C.K) + \frac{1}{2}(C.C)$. Now the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D+C) \longrightarrow \mathcal{O}_X(D+C)|_C \longrightarrow 0$$

gives

$$\chi(\mathcal{O}_X(D+C)) - \chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D+C)|_C),$$

which by Riemann–Roch for curves equals $\deg(\mathcal{O}_X(D+C)|_C) + 1 - g(C)$. By Lecture 19, Lemma 19.4.2, we have

$$\deg(\mathcal{O}_X(D+C)|_C) = ((D+C).C) = (D.C) + (C.C).$$

Moreover, by the adjunction formula (Lecture 17, Example 17.3.4) we have

$$2g(C) - 2 = \deg(\omega_C) = \deg(\omega_X(C)|_C) = (K.C) + (C.C),$$

and so $1 - g(C) = -\frac{1}{2}((K.C) + (C.C))$. Combining what we know yields the result. □

As in the case of curves, we define $\ell(D) = \dim H^0(X, \mathcal{O}_X(D))$. Riemann–Roch combined with Serre duality (in the form of Remark 20.1.2) gives the following inequality.

Corollary 20.2.3. *For any divisor D we have*

$$\ell(D) + \ell(K-D) \geq (D \cdot (D-K)) + \chi(\mathcal{O}_X).$$

In particular, if $(D.D) > 0$ then for $n \gg 0$, either nD or $K - nD$ is effective.

Proof. For the first statement note that

$$\ell(D) + \ell(K-D) = \dim H^0(X, \mathcal{O}_X(D)) + \dim H^2(X, \mathcal{O}_X(D)) \geq \chi(\mathcal{O}_X(D)) = (D \cdot (D-K)) + \chi(\mathcal{O}_X).$$

For the second, note that the right-hand side of the inequality for the divisor nD equals

$$n^2(D.D) - n(D.K) + \chi(\mathcal{O}_X)$$

which will be positive for $n \gg 0$. □

20.3. The Hodge index theorem

In the last lecture, we introduced the notion of an ample invertible sheaf. We call a divisor ample if the corresponding line bundle is ample. For the theorem below (along with its proof) it is important to remember the following:

(1) If $X \subseteq \mathbb{P}^n$ then the restriction of $\mathcal{O}(1)$ is ample. In other words, if $H \subseteq \mathbb{P}^n$ is a hyperplane which does not contain X, then $X \cap H$ is an ample divisor.

(2) If H is an ample divisor and C is an effective divisor, then $(H.C) > 0$. Moreover, the divisor nH is effective for $n \gg 0$, and consequently we have $(H.H) > 0$.

(3) If H is an ample divisor and C is any divisor, then for $n \gg 0$ the divisor $C + nH$ is ample, effective, and has $H^q(X, \mathcal{O}_X(C + nH)) = 0$ for $q > 0$.

Theorem 20.3.1 (Hodge index theorem)**.** *Let X be a smooth projective surface and let H be an ample divisor on X. Let D be another divisor, and suppose that $D \cdot H = 0$ and $D \cdot C \neq 0$ for some divisor C. Then $D^2 < 0$.*

Proof. Suppose first that $D^2 > 0$. Let $H' = D + mH$ for $m \gg 0$ such that H' is ample (Lecture 19, Lemma 19.2.2(f)). Then $H' \cdot D = D^2 + mD \cdot H > 0$. Therefore for $n \gg 0$ we have

$$H' \cdot (K - nD) = H' \cdot K - nD^2 < 0,$$

and hence $h^0(K - nD) = 0$. Therefore $h^2(nD) = 0$ for $n \gg 0$ by Serre duality.

The Riemann–Roch formula gives

$$h^0(nD) - h^1(nD) + h^2(nD) = \chi(\mathcal{O}_X(nD)) = nD(nD + K) + \chi(\mathcal{O}_X) = D^2 n^2 + O(n)$$

which is positive for $n \gg 0$. But since $h^2(nD) = 0$ for $n \gg 0$, we must have $h^0(nD) > 0$ for $n \gg 0$. But then $nD \cdot H > 0$ (being the intersection of an ample divisor with a non-trivial effective divisor), contradicting our assumption that $D \cdot H = 0$.

Suppose now that $D^2 = 0$. Let $C' = (H^2)C - (C \cdot H)H$, then $C' \cdot D = C \cdot D \neq 0$ but now also $C' \cdot H = 0$. Consider the family of divisors $nD + C'$. We have

$$(nD + C')^2 = n^2 D^2 + n(D \cdot C') + (C')^2 = n(D \cdot C') + (C')^2$$

which is a non-constant linear function. Therefore there exists an n such that $(nD + C')^2 > 0$. We can now apply the case $D^2 > 0$ to the divisor $nD + C'$ in place of D to get a contradiction. ☐

Definition 20.3.2 (Néron–Severi group)**.** Let X be a smooth projective surface. A divisor $D \subseteq X$ is **numerically trivial** if $D \cdot C = 0$ for every divisor C. We denote by $\mathrm{NS}(X)$ the quotient of $\mathrm{Pic}(X)$ by the subgroup consisting of numerically trivial divisors. We call it the **Néron–Severi group** of X. The intersection product induces a non-degenerate symmetric bilinear pairing on $\mathrm{NS}(X)$.

Theorem 20.3.3 (Theorem of the base, Néron)**.** *The group $\mathrm{NS}(X)$ is a free group of finite rank.*

Definition 20.3.4. The rank of $\mathrm{NS}(X)$ is called the **Picard rank** of X and is denoted by $\rho(X)$.

Corollary 20.3.5 (Equivalent form of the Hodge index theorem)**.** *Equip the real vector space $\mathrm{NS}(X)_{\mathbb{R}}$ with the intersection pairing. It has signature $(1, \rho(X) - 1)$.*

Remark 20.3.6 (Cones)**.** Various kinds of line bundles give rise to convex cones in the real vector space $\mathrm{NS}(X)_{\mathbb{R}}$. The **effective cone** $\mathrm{Eff}(X)$ is spanned by the classes of effective divisors. Property (2) of ample divisors implies that the **ample cone** is the interior of the dual cone $\mathrm{Eff}(X)^\vee = \mathrm{Nef}(X)$ called the **nef cone** ("nef" stands for "numerically effective"). Property (3) states that the ample cone is contained in the effective cone and it has non-empty interior, so that the ray $C + \mathbb{R}_{>0} \cdot H$ for ample H eventually intersects the ample cone. The structure of the nef cone is important in higher-dimensional birational geometry.

We shall later need the following less straightforward corollary of the Hodge index theorem.

Proposition 20.3.7. *Let C be a smooth projective curve, let $P \in C$, and let $X = C \times C$. Let $C_1 = \{P\} \times C$ and $C_2 = C \times \{P\}$. Then for any divisors D_1, D_2 on X we have*

$$\begin{aligned} |(C_1 \cdot D_1)(C_2 \cdot C_2) + (C_1 \cdot D_2)(C_2 \cdot D_1) - (D_1 \cdot D_2)| \leq & \sqrt{2(C_1 \cdot D_1)(C_2 \cdot D_1) - (D_1 \cdot D_1)} \\ & \cdot \sqrt{2(C_1 \cdot D_2)(C_2 \cdot D_2) - (D_2 \cdot D_2)}. \end{aligned}$$

Proof. We have $(C_1.C_2) = 1$ since the curves intersect transversally at the unique point (P,P). Next, we check that $(C_1.C_1) = 0$; indeed, this intersection number equals the degree of $\mathcal{O}(C_1)|_{C_1}$, but the sheaf $\mathcal{O}_X(C_1)|_{C_1}$ is free (spanned by the image of a uniformizer of $\mathcal{O}_{C,P}$) so its degree is zero. Similarly $(C_2.C_2) = 0$. We are now in position to apply Exercise 6 on Problem Set 5. □