5. Lecture 5 (Feb 5): Local rings, rational maps

Recommended reading: Hartshorne I.4

5.1. Local rings, function fields, and rational maps

Let X be an affine algebraic set with coordinate ring $A = \mathcal{O}(X)$. Recall that irreducible closed subsets $Z \subseteq X$ correspond to prime ideals $\mathfrak{p} \subseteq A$. If Z is an irreducible subset, then every non-empty open subset is dense, and the intersection of two non-empty opens is non-empty.

Definition 5.1.1. Let X be a (not necessarily affine) algebraic set and let $Z \subseteq X$ be an irreducible closed subset. We define the **stalk at** Z (also called the **local ring** at Z) as the filtered colimit

$$\mathcal{O}_{X,Z} = \varinjlim_{U \cap Z \neq \emptyset} \mathcal{O}(U)$$

over all open subsets $U \subseteq X$ which intersect Z (this is a filtered colimit since Z is irreducible, by the previous remark).

In plain terms, by the basic properties of filtered colimits, an element of $\mathcal{O}_{X,Z}$ is an equivalence class of pairs (U, f) where $U \subseteq X$ is an open intersecting Z and where $f \in \mathcal{O}(U)$, where we identify (U, f) with (U', f') if there exists an open $U'' \subseteq U \cap U'$ intersecting Z such that $f|_{U''} = f'|_{U''}$.

Examples 5.1.2. (a) Let $x \in X$. Then $\{x\} \subseteq X$ is closed and irreducible, and we denote the ring $\mathcal{O}_{X,\{x\}}$ more simply by $\mathcal{O}_{X,x}$. Its elements are **germs** of regular functions defined in a neighborhood of x.

(b) At the other extreme, suppose that X is itself irreducible (i.e. a "variety"). In this case we can take $Z = X$. We denote the ring $\mathcal{O}_{X,X}$ more simply by $k(X)$ and call it the **function field** of X. Its elements are represented by regular functions defined on a non-empty open subset of X.

Remark 5.1.3 (From the future)**.** We shall later notice that $\mathcal{O}_{X,x}$ is simply the *stalk* of the structure sheaf $\mathcal{O}_X$ (the assignment $U \mapsto \mathcal{O}(U)$). Moreover, scheme theory extends this interpretation to $\mathcal{O}_{X,Z}$ for an arbitrary closed irreducible Z. Namely, the scheme X^{sch} corresponding to X adds a unique generic point η_Z for every closed irreducible $Z \subseteq X$. Then $\mathcal{O}_{X,Z} \simeq \mathcal{O}_{X^{\mathrm{sch}},\eta_Z}$.

Lemma 5.1.4. *Let X be an algebraic set and let $Z \subseteq X$ be an irreducible closed subset.*

(a) Let $U \subseteq X$ be an affine open intersecting Z, so that $Z \cap U$ is an irreducible closed subset of U. Let $A = \mathcal{O}(U)$ and let $\mathfrak{p} = \mathcal{I}(Z \cap U) \subseteq A$ be the prime ideal corresponding to $Z \cap U$. Then

$$\mathcal{O}_{X,Z} = \mathcal{O}_{U,U \cap Z} = A_{\mathfrak{p}}.$$

(Recall that $A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$.) In particular, $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal consisting of germs of functions (U, f) which vanish on $Z \cap U$.

(b) If X is irreducible, then $k(X)$ is a field, equal to the fraction field of $\mathcal{O}(U)$ (which is a domain) for every non-empty affine open $U \subseteq X$.

(c) The residue field of the local ring $\mathcal{O}_{X,Z}$ is the function field $k(Z)$ of Z.

Proof. (a) We first notice that if $U \subseteq X$ is any open intersecting Z, then $\mathcal{O}_{X,Z} \simeq \mathcal{O}_{U,U\cap Z}$. This follows from abstract properties of filtered colimits (those $V \subseteq X$ which are contained in U are cofinal among those which intersect Z), and is also easy to show directly. Namely, if (V, f) is an element of $\mathcal{O}_{X,Z}$, we get a corresponding element $(V \cap U, f|_{V \cap U})$ of $\mathcal{O}_{U, Z \cap U}$, and if (V, f) is an element of $\mathcal{O}_{U, U \cap Z}$, then (V, f) also represents an element of $\mathcal{O}_{X,Z}$. We verify immediately that these correspondences give well-defined mutually inverse ring homomorphisms.

To obtain the rest, we may assume that $X = U$ is affine. Recall that in this case X has a base of the topology consisting of standard opens $D(f)$ for $f \in A$. Therefore, if $V \subseteq X$ is an open intersecting Z, then there exists an $f \in A$ such that $D(f) \subseteq V$ and $D(f) \cap Z \neq \emptyset$. In defining $\mathcal{O}_{X,Z}$, we can thus restrict to a filtered colimit over opens only of this special form. Notice that the last condition $D(f) \cap Z \neq \emptyset$ is equivalent to $f \notin \mathfrak{p}$. Moreover, we have proved that $\mathcal{O}(D(f)) = A[f^{-1}]$. Assembling these observations, we get

$$\mathcal{O}_{X,Z} = \varinjlim_{D(f) \cap Z \neq \emptyset} \mathcal{O}(D(f)) = \varinjlim_{D(f) \cap Z \neq \emptyset} A[f^{-1}] = A_{\mathfrak{p}}.$$

As shown in commutative algebra, this is a local ring with maximal ideal $\mathfrak{p} \cdot A_{\mathfrak{p}}$, which coincides with those $f \in \mathcal{O}_{X,Z}$ which vanish on Z.

(b) Apply (a) to $X = Z$, in which case $\mathfrak{p} = (0)$ and $A_{\mathfrak{p}}$ is the fraction field of A.

(c) Let $U \subseteq X$ be an affine open intersecting Z and let $A = \mathcal{O}(U)$. Recall that the residue field $A_{\mathfrak{p}}/\mathfrak{p} \cdot A_{\mathfrak{p}}$ of $\mathcal{O}_{X,Z} = A_{\mathfrak{p}}$ (denoted by $\kappa(\mathfrak{p})$ in commutative algebra) can also be expressed as the fraction field of $A/\mathfrak{p}$. Now $A/\mathfrak{p} = \mathcal{O}(U \cap Z)$, so the result follows from (b) applied to $U \cap Z$. □

Remark 5.1.5 (Direct proof of a part of (a) for swf's)**.** The fact that $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal consisting of functions vanishing on Z holds more generally for any swf X and a closed irreducible $Z \subseteq X$. We give the direct argument here since it illustrates well the motivation behind the second axiom of an swf.

Let $\mathfrak{m} \subseteq \mathcal{O}_{X,Z}$ be the set of all $(U, f) \in \mathcal{O}_{X,Z}$ such that $f|_{U \cap Z} = 0$. We verify immediately that (1) this condition does not depend on the choice of representative (this uses Z irreducible), and (2) is an ideal of $\mathcal{O}_{X,Z}$.

In order to check that $\mathcal{O}_{X,Z}$ is local with maximal ideal $\mathfrak{m}$, we need to show that every (U, f) which does not belong to $\mathfrak{m}$ is an invertible element of $\mathcal{O}_{X,Z}$. The condition that $(U, f) \notin \mathfrak{m}$ means precisely that $D(f)$ meets Z. By axiom (2) of the definition of a space with functions, this set is open and $1/f \in \mathcal{O}(D(f))$. Thus $(U, f) \cdot (D(f), 1/f) = 1$ in $\mathcal{O}_{X,Z}$, and (U, f) is invertible.

Lemma 5.1.6. *Let $\phi \colon Y \to X$ be a morphism between algebraic sets and let $W \subseteq Y$ be an irreducible closed subset. Let $Z \subseteq X$ be the closure of $\phi(Z)$. Then Z is also irreducible, and f induces a local homomorphism of local rings*

$$\phi^* \colon \mathcal{O}_{X,Z} \longrightarrow \mathcal{O}_{Y,W}.$$

Proof. That $\phi(W)$ and its closure Z are both irreducible is easy general topology. Moreover, $\phi(W)$ is dense in Z by definition. Therefore, if $U \subseteq X$ is an open intersecting Z, it has to intersect $f(Z)$, and thus $\phi^{-1}(U) \subseteq Y$ is an open intersecting W. The pull-back maps $\phi^* \colon \mathcal{O}_X(U) \to \mathcal{O}_Y(\phi^{-1}(U))$ for varying U induce map on filtered colimits

$$\phi^* \colon \mathcal{O}_{X,Z} = \varinjlim_{U} \mathcal{O}_X(U) \longrightarrow \varinjlim_{U} \mathcal{O}_Y(\phi^{-1}(U))$$

(both colimits over opens $U \subseteq X$ meeting Z) which we compose with the natural map (from the universal property of direct limit!)

$$\varinjlim_{U} \mathcal{O}_Y(\phi^{-1}(U)) \longrightarrow \varinjlim_{V} \mathcal{O}_Y(V) = \mathcal{O}_{Y,W}$$

(second colimit over opens $V \subseteq Y$ meeting W) to obtain the desired map $\phi^* \colon \mathcal{O}_{X,Z} \to \mathcal{O}_{Y,W}$. This homomorphism is local thanks to Lemma 5.1.4(a): if $f \in \mathcal{O}_X(U)$ vanishes on $U \cap Z$ then $\phi^*(f) = f \circ \phi \in \mathcal{O}_Y(\phi^{-1}(U))$ vanishes on $\phi^{-1}(U \cap Z) \supseteq \phi^{-1}(U) \cap W$. □

We shall now consider rational maps between varieties. For this, let us note the following straightforward corollary of Lemma 5.1.6. For this, let us call a map $\phi\colon Y \to X$ **dominant** if $\phi(Y)$ is dense in X (and hence, by Chevalley's theorem, contains a dense open subset of X).

Corollary 5.1.7. *A dominant map between varieties $\phi\colon Y \to X$ induces an extension of function fields $\phi^*\colon k(X) \hookrightarrow k(Y)$.*

Proof. Apply Lemma 5.1.6 to $W = Y$, so $\mathcal{O}_{Y,W} = k(Y)$. Since ϕ is dominant, $\phi(W) = \phi(Y)$ is dense, and we have $Z = X$, so $\mathcal{O}_{X,Z} = k(X)$. ☐

A rational map is a germ of a function between varieties.

Definition 5.1.8. Let X and Y be varieties (i.e. irreducible algebraic sets). A **rational map** from Y to X is an equivalence class of pairs (U, f) where $U \subseteq Y$ is a non-empty open subset and where $f\colon U \to X$ is a map of varieties, where we identify (U, f) and (U', f') if $f = f'$ on some non-empty open $U'' \subseteq U \cap U'$. We call a rational map (U, f) **dominant** if $f(U)$ is dense in X (this condition depends only on the equivalence class of (U, f)).

Remark 5.1.9.

(a) A dominant rational map (U, f) from Y to X induces a pull-back map $f^*\colon k(X) \to k(Y)$ between the function fields.

(b) Conversely, let $k(X) \to k(Y)$ be a map of k-algebras. Then there exists a unique dominant rational map $Y \to X$ inducing this field extension.

(c) Dominant rational maps can be composed. The resulting category of varieties and rational maps is equivalent to the opposite of the category of finitely generated field extensions of k.

(d) If $V \subseteq Y$ is an open such that a given rational map from Y to X is represented by a pair (V, f), we say that f is **defined on** V. If Y is separated, there exists a largest open $V \subseteq Y$ on which a given rational map is defined.

Definition 5.1.10. A rational map f from Y to X is **birational** if it is dominant and if it admits an inverse (in the category of dominant rational maps), or equivalently if there exist non-empty opens $V \subseteq Y$ and $U \subseteq X$ such that f induces an isomorphism $V \simeq U$, or equivalently if it induces an isomorphism $k(X) \simeq k(Y)$. We say that two varieties X and Y are **birational** if there exists a birational rational map from Y to X. We say that a variety X is **rational** if it is birational to $\mathbb{P}^n$ for some $n \geq 0$.

Example 5.1.11. We shall prove later that the cubic plane curve $V(Y^2 - X^3 - X)$ is not rational.