

Algebraic geometry

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Preface

These lecture notes contain the material covered in a lecture course taught at KSE in Spring 2026. The key prerequisite for the course is a basic knowledge of commutative algebra (including the prime spectrum of a ring) and some sheaf theory.

Algebraic geometry is the study of algebraically defined geometric objects, such as algebraic varieties (roughly, zero sets of systems of polynomial equations in many variables, over an algebraically closed field) or schemes (spaces which locally look like the prime spectrum of a commutative ring). It has a close relationship with commutative algebra, differential geometry, complex analysis, number theory, and representation theory. The goal of the course is to introduce the audience to the basic objects of study in algebraic geometry: algebraic varieties and schemes, and (sheaves of) modules over them. The last lecture showcases the power of the theory thus developed by giving a proof of the Weil conjectures (Riemann hypothesis) for curves over a finite field.

The notes are in several places incomplete and might be updated in the future. **Orange text** and orange boxes ■ indicate things left to be filled in. Comments (including typos) are very welcome.

1. Lecture 1 (Jan 13): Affine algebraic sets

Course info

- shelter 8 (building across the street; follow the crowd)
- course website: <https://achinger.impan.pl/ag2026.html>
- email: pachinger@impan.pl
- Moodle, Slack, Zoom, Google Drive
- weekly homework posted on the course website, submit solutions by email with filename Lastname-N.pdf where N is the number of the problem set
- extra credit: problems marked with an asterisk (due end of term), short (3–5 pages) term papers on a topic of your choice (more info soon)
- reading week (no class): Apr 6–10, exam (written or oral) in the last week (Apr 14 or 16)
- office hours: Thursdays 10am (email me if you plan to come)
- I might post some lecture notes (in fact, I am just doing that)
- literature (see Google Drive):
 1. [Kem93] G. Kempf *Algebraic Varieties* (excellent for a one semester course)
 2. [Kem93] R. Hartshorne *Algebraic Geometry* (a bit heavy for us, but we will read parts of it)
 3. [Vak25] R. Vakil *The Rising Sea: Foundations of Algebraic Geometry*
 4. [Rei88] M. Reid *Undergraduate Algebraic Geometry*
 5. [Mum99] D. Mumford *The Red Book of Varieties and Schemes*

1.1. Affine algebraic sets and their k -points

Recommended reading for this lecture: [Har77, I.1] (and bits of I.2).

Algebraic geometry studies algebraically defined geometric objects, of which the most basic are **(affine) algebraic sets**. Fix a field k and consider a system of polynomial equations in n variables

$$X: \begin{cases} f_1(T_1, \dots, T_n) = 0 \\ \dots \\ f_r(T_1, \dots, T_n) = 0 \end{cases} \quad (1.1.1)$$

where $f_1, \dots, f_r \in k[T_1, \dots, T_n]$. Importantly, our basic object X is this system, not its **set of solutions** in k^n , which we denote by $X(k)$:

$$X(k) = \{(x_1, \dots, x_n) \in k^n : f_i(x_1, \dots, x_n) = 0 \text{ for } i = 1, \dots, r\} \subseteq k^n.$$

More generally, if K is a field containing k (or just a k -algebra), we can define $X(K) \subseteq K^n$ as the set of solutions of (1.1.1) in K^n . We also define the **coordinate ring** of X as the k -algebra

$$A = \mathcal{O}(X) = k[T_1, \dots, T_n]/(f_1, \dots, f_r).$$

Examples 1.1.1. The most basic examples of systems of polynomial equations:

- (a) If $r = 0$ (i.e. n variables and no equations), we call the system the **affine n -space** and denote it by \mathbb{A}^n . We have $\mathbb{A}^n(K) = K^n$ and $\mathcal{O}(\mathbb{A}^n) = k[T_1, \dots, T_n]$. For a system X with n variables, we write $X \subseteq \mathbb{A}^n$ to signify that $X(K) \subseteq K^n$.
- (b) A **hypersurface** is the system consisting of a single equation $f(T_1, \dots, T_n) = 0$, where $f \in k[T_1, \dots, T_n]$ is a non-constant polynomial (often assumed to be irreducible). If $\deg(f) = 1, 2, 3, 4, 5, 6, 7, 8$ we call f a hyperplane, a quadric, a cubic, a quartic, a quintic, a sextic, a septic, an octic.
- (c) A **plane curve** is a hypersurface in \mathbb{A}^2 , i.e. a system C with a single equation

$$f(X, Y) = 0$$

where $f \in k[X, Y]$ is a non-constant polynomial. For example, the *lemniscate of Bernoulli* is defined by the equation

$$(X^2 + Y^2)^2 = X^2 - Y^2.$$

- (d) If $\deg(f) = 2$ and f is irreducible, we call C a **conic**. That is, a conic is a quadric hypersurface in \mathbb{A}^2 .
- (e) Consider $k = \mathbb{R}$ and the conic C defined by

$$X^2 + Y^2 = -1.$$

We have $C(\mathbb{R}) = \emptyset$. However, $\mathcal{O}(C) \neq 0$ and $C(\mathbb{C}) \neq \emptyset$.

- (f) The “fat point” $X \subseteq \mathbb{A}^1$ defined by the single equation

$$T^2 = 0.$$

We have $\mathcal{O}(X) = k[T]/(T^2)$, which is non-reduced (has a nonzero nilpotent element, namely T). For any field K , we have $X(K) = \{0\}$, which is the same as for the equation $X' : T = 0$. We need more advanced technology (schemes) to distinguish between the geometric objects X and X' . For now though we shall mostly stick to systems of equations which give reduced k -algebras.

We note that the algebra $A = \mathcal{O}(X)$ “remembers” the set $X(K)$, namely we have a bijection

$$X(K) = \text{Hom}_k(A, K)$$

between the set of K -valued solutions and the set of k -algebra homomorphisms from A to K . Indeed, giving a map $\phi : k[T_1, \dots, T_n] \rightarrow K$ is the same as giving its values $x_i = \phi(T_i)$ i.e. an element $(x_1, \dots, x_n) \in K^n$. Such a map factors (uniquely) through $A = k[T_1, \dots, T_n]/(f_1, \dots, f_r)$ if and only if $\phi(f_j) = 0$. But

$$\phi(f_j(T_1, \dots, T_n)) = f_j(\phi(T_1), \dots, \phi(T_n)) = f_j(x_1, \dots, x_n)$$

which happens precisely when $(x_1, \dots, x_n) \in X(K)$.

Thus, a system X (1.1.1) determines A which determines the solution set $X(k)$. We regard X as too rigid, the set $X(k)$ as too primitive for describing a geometric object (for example, it could be empty), and the algebra A as just right.

The distinction between X , A , and $X(k)$ becomes less serious when k is algebraically closed, and indeed most of the methods of algebraic geometry are developed over $k = \bar{k}$, even if the motivation is the study of $X(k)$ for $k = \mathbb{R}, \mathbb{Q}$, or a finite field. Life is easier over an algebraically closed field k thanks to Hilbert’s Nullstellensatz, which says that the only reason for the system (1.1.1) to have no solutions in k^n is that we can algebraically rearrange the equations to obtain the equation $1 = 0$:

Theorem 1.1.2 (Hilbert's Nullstellensatz). *Consider a system of polynomial equations (1.1.1) over a field k . The following are equivalent:*

- (a) $X(K) = \emptyset$ for every field K containing k ;
- (b) $X(\bar{k}) = \emptyset$, where \bar{k} is an algebraic closure of k ;
- (c) there exist polynomials $h_1, \dots, h_r \in k[T_1, \dots, T_n]$ such that

$$1 = h_1 f_1 + \dots + h_r f_r.$$

- (d) $\mathcal{O}(X) = 0$.

Proof. The implications (a) \Rightarrow (b), (c) \Leftrightarrow (d) \Rightarrow (a) are obvious. We shall deduce the remaining (b) \Rightarrow (d) from the following theorem, another version of Nullstellensatz.

Theorem 1.1.3 (Basic form of Nullstellensatz). *Let k be a field and let K be a finitely generated k -algebra which is a field. Then K is a finite extension of k .*

Last term, which we proved it using the Artin–Tate lemma. Just for fun, let us give a simple proof in case k is uncountable (e.g. $k = \mathbb{C}$). Note first that it suffices to show that every $x \in K$ is algebraic over k (as a finitely generated algebraic extension is finite). Suppose $x \in K$ is not algebraic over k , then we have an injection $k(T) \subseteq K$. Now, we get a contradiction because

- $k(T)$ has uncountable dimension over k , as the elements $1/(T - \alpha)$ are linearly independent over k for $\alpha \in k$ and k is uncountable;
- K has countable dimension over k , since being finitely generated over k it is a quotient of $k[T_1, \dots, T_n]$ for some $n \geq 0$, and this space has a countable basis consisting of all monomials in the T_i .

To show (b) \Rightarrow (d), suppose $A = \mathcal{O}(X) \neq 0$. Thus A has a maximal ideal \mathfrak{m} , and $K = A/\mathfrak{m}$ is a field which is generated over k by images of the T_i . By the above theorem, it is finite over k , and hence we can find an embedding $K \subseteq \bar{k}$, so that $X(K) \subseteq X(\bar{k})$. Now, by construction $X(K) = \text{Hom}_k(A, K)$ is non-empty (because we have the quotient map $A \rightarrow A/\mathfrak{m} = K$), and hence $X(\bar{k})$ is non-empty. \square

From now on we shall assume that k is algebraically closed.

This assumption implies in particular that the set $X(k) = \text{Hom}_k(A, k)$ coincides with the set $\text{MSpec}(A)$ of maximal ideals of A . I will also use \mathbb{A}^n to mean $\mathbb{A}^n(k) = k^n$.

Definition 1.1.4. Let $P = k[T_1, \dots, T_n]$ be the polynomial ring.

- (1) For $f \in P$, we write

$$V(f) = \{(x_1, \dots, x_n) \in k^n : f(x_1, \dots, x_n) = 0\}$$

and $D(f) = k^n \setminus V(f)$.

- (2) For an ideal $I \subseteq P$, we write $V(I) = \bigcap_{f \in I} V(f)$. (Note that we do not define $D(I)$.)
- (3) An **affine algebraic set** is a subset $Z \subseteq k^n$ of the form $V(I)$ for some $I \subseteq P$.
- (4) For a subset $Z \subseteq k^n$, we denote by $\mathcal{J}(Z) \subseteq P$ the ideal

$$\mathcal{J}(Z) = \{f \in P : f(x_1, \dots, x_n) = 0 \text{ for all } x \in Z\}.$$

Note that for $f_1, \dots, f_r \in P$ and $I = (f_1, \dots, f_r)$ we have $V(I) = V(f_1) \cap \dots \cap V(f_r)$.

Proposition 1.1.5. Let $P = k[T_1, \dots, T_n]$ be the polynomial ring.

- (a) For any family of ideals $I_\alpha \subseteq P$ we have $\bigcap V(I_\alpha) = V(\sum I_\alpha)$.
- (b) For two ideals $I, J \subseteq P$ we have $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.
- (c) Affine algebraic sets are the closed sets for a topology on k^n (called the Zariski topology).
- (d) The open sets $D(f)$ form a base for the topology on k^n closed under pairwise intersection: $D(f) \cap D(g) = D(fg)$.
- (e) For every subset $Z \subseteq k^n$, the ideal $\mathcal{J}(Z)$ is radical and $V(\mathcal{J}(Z)) = \bar{Z}$ (the closure of Z).
- (f) For every ideal $I \subseteq P$ we have $\mathcal{J}(V(I)) = \sqrt{I}$ (the radical of I).
- (g) The maps V and \mathcal{J} establish mutually inverse bijections between closed subsets of k^n and radical ideals of P .

Proof. Everything is easy to show except for the containment $\mathcal{J}(V(I)) \subseteq \sqrt{I}$ in (f), another form of the Nullstellensatz. Let us deduce it from Theorem 1.1.2. Write $I = (f_1, \dots, f_r)$. We must show that if $g \in P$ vanishes on $V(f_1, \dots, f_r)$ then $g^n \in I$ for some $n \geq 1$. Equivalently, g is nilpotent in $A = P/I$. Consider the system of equations in $n+1$ variables T_1, \dots, T_n, T_{n+1} :

$$X' = \begin{cases} f_1(T_1, \dots, T_n) = 0 \\ \dots \\ f_r(T_1, \dots, T_n) = 0 \\ g(T_1, \dots, T_n) \cdot T_{n+1} - 1 = 0. \end{cases}$$

Then the assumption $V(g) \supseteq V(f_1, \dots, f_r)$ is equivalent to $X'(k) = \emptyset$. Thus by Theorem 1.1.2 we have $A' = 0$ where $A' = \mathcal{O}(X')$. However

$$A' = \left(\frac{k[T_1, \dots, T_n]}{(f_1, \dots, f_r)} \right) [T_{n+1}] / (gT_{n+1} - 1) = A[g^{-1}],$$

the localization of A at g . Then $A' = 0$ means that $0/1 = 1/1$ in $A[g^{-1}]$, which by definition of localization means that $g^n = 0$ in A , and we are done. \square

Note that the last result implies that we can recover A up to nilpotents from the set $X(k) \subseteq k^n$:

Corollary 1.1.6. We have $A/\sqrt{0} \simeq P/\mathcal{J}(X(k))$.

Examples 1.1.7. (a) A proper subset of $\mathbb{A}^1 = k$ is closed if and only if it is finite.

- (b) Let $Z \subseteq \mathbb{A}^2$ be a proper closed subset. Then Z is the union of a plane curve $f(X, Y) = 0$ and a finite set. (We will prove this later.)

1.2. Projective algebraic sets

The multiplicative group k^\times acts freely on the open subset $k^{n+1} \setminus \{0\}$ of k^{n+1} by

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n).$$

We define the **projective n -space** to be the quotient (orbit space)

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times.$$

We denote by $\pi: k^{n+1} \setminus 0 \rightarrow \mathbb{P}^n(k)$ the quotient map, and by $(x_0 : \dots : x_n)$ the image of $(x_0, \dots, x_n) \in k^{n+1} \setminus 0$.

Let $P = k[T_0, \dots, T_n]$. For $(x_0 : \dots : x_n) \in \mathbb{P}^n(k)$, the expression

$$f(x_0, \dots, x_n)$$

does not make sense. However, if f is *homogeneous*, the condition

$$f(x_0, \dots, x_n) = 0$$

makes sense (i.e. is independent of the choice of a representative (x_0, \dots, x_n)). Recall that a polynomial $f \in P$ is **homogeneous of degree d** if we have an equality of polynomials in $P[\lambda]$

$$f(\lambda T_0, \dots, \lambda T_n) = \lambda^d f(T_0, \dots, T_n),$$

or equivalently if all monomials in f are of the same degree d . We say that f is homogeneous if it is homogeneous of degree d for some $d \geq 0$. If $P_d \subseteq P$ is the subspace of homogeneous polynomials of degree d , then

$$P = \bigoplus_{d \geq 0} P_d.$$

An ideal $I \subseteq P$ is **homogeneous** if it is generated by a set of homogeneous elements, or equivalently if

$$I = \bigoplus_{d \geq 0} (I \cap P_d).$$

Lemma 1.2.1. *Let $Z \subseteq k^{n+1}$ be a closed subset. The following are equivalent:*

- (a) Z is **conical**, i.e. invariant under the action of k^\times ;
- (b) $Z = V(I)$ for a homogeneous ideal $I \subseteq P$;
- (c) the ideal $\mathcal{J}(Z) \subseteq P$ is homogeneous.

Proof. The implications (c) \Rightarrow (b) \Rightarrow (a) are straightforward. To show (a) \Rightarrow (c), let $f \in \mathcal{J}(Z)$ and write $f = \sum f_d$ where $f_d \in P_d$. We must show that all f_d belong to $\mathcal{J}(Z)$ i.e. vanish on Z . Let $(x_0, \dots, x_n) \in Z \setminus 0$, and consider the function $\phi: k \rightarrow k$ given by

$$\phi(\lambda) = f(\lambda x_0, \dots, \lambda x_n) = \sum_{d \geq 0} f_d(\lambda x_0, \dots, \lambda x_n) = \sum_{d \geq 0} f_d(x_0, \dots, x_n) \lambda^d.$$

Since Z is conical, we have $(\lambda x_0, \dots, \lambda x_n) \in Z$, and hence $\phi(\lambda)$ is identically zero. Since it is a polynomial in λ by the above expression, all of its coefficients are zero, so $f_d(x_0, \dots, x_n) = 0$ for all d . \square

Definition 1.2.2. For a homogeneous ideal $I \subseteq P$, let

$$V_{\mathbb{P}}(I) = \{(x_0 : \dots : x_n) : f(x_0, \dots, x_n) = 0 \text{ for all homogeneous } f \in I\} = \pi(V(I) \setminus 0) \subseteq \mathbb{P}^n(k).$$

A **projective algebraic set** is a subset of $\mathbb{P}^n(k)$ of the form $V_{\mathbb{P}}(I)$ for some homogeneous ideal $I \subseteq P$.

As in the affine case, projective algebraic sets are the closed sets of a topology on $\mathbb{P}^n(k)$ called the **Zariski topology**, with basis of open sets given by

$$D_{\mathbb{P}}(f) = \{(x_0 : \dots : x_n) : f(x_0, \dots, x_n) \neq 0\} = \pi(D(f))$$

for homogeneous $f \in P$.

Let us show how $\mathbb{P}^n(k)$ can be expressed as the union of $n + 1$ copies of \mathbb{A}^n . Let

$$U_i = D_{\mathbb{P}}(T_i) = \{(x_0 : \cdots : x_n) : x_i \neq 0\} \subseteq \mathbb{P}^n(k)$$

$$V_i = \{(x_0, \dots, x_n) \in k^{n+1} : x_i = 1\} \subseteq k^{n+1} \setminus 0.$$

Then $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$ and for each i , the restriction of π to V_i gives a homeomorphism $V_i \simeq U_i$, with inverse given by

$$(x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i} : \cdots : \frac{x_n}{x_i} \right).$$

Example 1.2.3 (Projective line). Let us explicate the above description for $n = 1$. The projective line \mathbb{P}^1 is the union of $U_0 = 1 \times k \simeq \mathbb{A}^1$ with coordinate x_1 and $U_1 = k \times 1 \simeq \mathbb{A}^1$ with coordinate x_0 . We have $U_0 \cap U_1 \simeq k^\times$ with coordinate $x_1 = x_0^{-1}$. We can write

$$\mathbb{P}^1(k) = \mathbb{A}^1 \sqcup \{\infty\}, \quad \infty = (0 : 1).$$

Example 1.2.4 (Projective plane). Consider $n = 2$ and let us name the coordinates T_0, T_1, T_2 by X, Y, Z . Let $U = U_2 = \{Z \neq 1\} = \mathbb{A}^2$ with coordinates $x = X/Z$ and $y = Y/Z$. Then $\mathbb{P}^2 \setminus U = V(Z)$ can be identified with \mathbb{P}^1 with homogeneous coordinates $(x : y)$.

The following lemma allows us to compute the projective closure of an affine hypersurface. For a polynomial $f \in k[T_1, \dots, T_n]$ (no T_0) of degree $d \geq 0$, write $f = \sum_{e=0}^d f_e$ where f_e is homogeneous of degree e , and let us define its **homogenization** as

$$\bar{f} = \sum_{e=0}^d T_0^{d-e} f_e \in P_d.$$

This is the unique homogeneous polynomial of degree d satisfying

$$f(T_1, \dots, T_n) = \bar{f}(1, T_1, \dots, T_n).$$

For example,

$$f = T_1^2 - T_2^3 - T_2 \quad \Rightarrow \quad \bar{f} = T_0 T_1^2 - T_2^3 - T_0^2 T_2.$$

Lemma 1.2.5. Let $f \in k[T_1, \dots, T_n]$ be a nonzero polynomial with homogenization $\bar{f} \in k[T_0, \dots, T_n]$. Then the closure of $V(f) \subseteq U_0 = k^n$ in $\mathbb{P}^n(k)$ is given by $V_{\mathbb{P}}(\bar{f})$.

We didn't prove this lemma on Jan 13. We give a proof in the subsequent subsection.

1.3. Problem session

We discussed charts on the projective space. We also introduced the following notion (used in problem 6).

Definition 1.3.1. Let $f \in k[X, Y]$ be a square-free nonconstant polynomial, defining an affine curve $C = V(f) \subseteq k^2$. We say that a point $P = (x, y) \in C$ is **singular** if $\partial f / \partial X$ and $\partial f / \partial Y$ both vanish at P . Otherwise, we say that P is a **nonsingular** or **smooth** point of C .

We mentioned, but didn't prove, the fact that every curve has only finitely many singular points.

List of problems:

(a) Show $(a) \Rightarrow (c)$ in Lemma 1.2.1

Solution. See the proof of Lemma 1.2.1.

- (b) Find a non-homogeneous ideal $I \subseteq k[T_0, \dots, T_n]$ whose zero set $Z = V(I) \subseteq k^{n+1}$ is conical.

Solution. Consider $I = (X^2 + Y, Y^2) \subseteq k[X, Y]$. This ideal is not homogeneous since $X^2 + Y \in I$ but $Y \notin I$. However, its zero set is $\{(0, 0)\}$ (note $\sqrt{I} = (X, Y)$), which is conical.

- (c) Find the points at infinity of the affine plane curves $Y = X^2$ and $XY = 1$.

Solution. Assuming Lemma 1.2.5 (problem 5 below), the closures of $V(Y - X^2)$ and $V(XY - 1)$ are cut out by the homogenized equations $YZ = X^2$ and $XY = Z^2$. Setting $Z = 0$ we get $0 = X^2$ and $XY = 0$. Thus, the points at infinity are $(0 : 1 : 0)$ (with multiplicity two) in the first example and $\{(0 : 1 : 0), (1 : 0 : 0)\}$ (corresponding to the vertical and horizontal asymptote) in the second example.

- (d) Prove that the Zariski topology on $k^2 = k \times k$ is not the product topology (with both factors given the Zariski topology).

Solution. The diagonal $V(X - Y) \subseteq k^2$ is closed in the Zariski topology, but not in the product topology. In fact, a proper subset $Z \subseteq k^2$ is closed in the product topology if and only if it is a finite union of horizontal lines, vertical lines, and points (straightforward, details omitted).

Klymentii asked if we can show that the two spaces are not homeomorphic (possibly by a map which is not the identity). The answer seems to be no, but to handle this we need to know something about irreducible closed subsets (next lecture). More precisely, irreducible closed subsets of $k \times k$ are precisely the vertical lines, horizontal lines, points, and the entire space. Thus the intersection of two distinct proper irreducible closed subsets has at most one point. However, in k^2 the subsets $V(Y)$ and $V(Y - X(X - 1))$ are irreducible and have exactly two points in common.

Bonus question: show that \mathbb{A}^2 and \mathbb{P}^2 are not homeomorphic. See also Problem 6* on Problem Set 1.

- (e) Prove Lemma 1.2.5

We didn't solve this problem. I added a proof in the next subsection.

- (f) (Hartshorne (I, Ex. 5.1)) Find the singular points of the following curves (assuming $\text{char}(k) \neq 2$).

- (a) $X^2 = X^4 + Y^4$;
 (b) $XY = X^6 + Y^6$;
 (c) $X^3 = Y^2 + X^4 + Y^4$;
 (d) $X^2Y + XY^2 = X^4 + Y^4$.

See Hartshorne's book for pictures of these singularities.

Solution. (c) Hands-on computation gives that $(0, 0)$ is the only singular point.

1.4. Bonus: Computing the closure

The following lemma allows us to compute the closure of a locally closed subset of \mathbb{A}^n .

Lemma 1.4.1. Let $I \subseteq P = k[T_1, \dots, T_n]$ be an ideal and let $g \in P$. Let

$$W = V(I) \subseteq \mathbb{A}^n \quad \text{and} \quad U = D(g) \subseteq \mathbb{A}^n,$$

and let $W' \subseteq \mathbb{A}^n$ be the closure of $W \cap U$. Consider the ideal

$$I' = \ker(P \rightarrow (P/I)[g^{-1}]) = \{f \in P : g^n f \in I \text{ for some } n \geq 1\}.$$

Then $W' = V(I')$. Moreover, if I is radical, then so is I' .

Proof. First, we show the equality between the two given definitions of I' . This follows from the general fact that for an element g of a ring B the kernel of the localization $B \rightarrow B[g^{-1}]$ consists of all $f \in B$ such that $g^n f = 0$ for some $n \geq 0$.

Now, the closure of $W \cap U$ is the intersection of all $V(f)$ for $f \in P$ which vanish on $W \cap U$. Let $f \in P$ be such an element. As in the proof of the variant of Nullstellensatz in Proposition 1.1.5, we consider the zero set

$$W'' = V(I, gT_{n+1} - 1) \subseteq \mathbb{A}^{n+1}.$$

The corresponding coordinate ring is

$$\mathcal{O}(W'') = k[T_1, \dots, T_n, T_{n+1}]/(I, gT_{n+1} - 1) = (P/I)[g^{-1}].$$

Then f (treated as an element of $P[T_{n+1}] = k[T_1, \dots, T_n, T_{n+1}]$) vanishes on W'' , and hence (by Proposition 1.1.5(f)) its image in $\mathcal{O}(W'') = (P/I)[g^{-1}]$ is nilpotent. This means that $f^m g^n \in I$ for some $n \gg 0$, in other words $f \in \sqrt{I'}$. Thus $\overline{W \cap U} = V(\sqrt{I'}) = V(I')$, and we are done. \square

The ideal I' in Lemma 1.4.1 is sometimes called the **g -saturation** of I . Using Lemma 1.4.1 we can now give a proof of Lemma 1.2.5.

Proof of Lemma 1.2.5. Let W be the closure of $V(f)$ and let $V = V_{\mathbb{P}}(\bar{f})$. We want to show that $W = V$, and since $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$, it suffices to check that $W \cap U_i = V \cap U_i$ for each i . For $i = 0$ this is clear, as $W \cap U_0 = V(f) = V \cap U_0$. Permuting the variables T_1, \dots, T_n , without loss of generality it suffices to consider the case $i = n$ (which saves us a minor annoyance with indices). The set U_n has coordinates (t_0, \dots, t_{n-1}) where $t_i = T_i/T_n$, and $U_0 \cap U_n$ is the set $D(t_0)$. Then $V(f) \cap U_n$ is

$$\{(t_0, \dots, t_{n-1}) : t_0 \neq 0 \text{ \& } f(t_1/t_0, \dots, t_{n-1}/t_0, 1/t_0) = 0\}.$$

Note that

$$t_0^d f(t_1/t_0, \dots, t_{n-1}/t_0, 1/t_0) = \bar{f}(t_0, t_1, \dots, t_{n-1}, 1).$$

Call this element $f' = \bar{f}(t_0, t_1, \dots, t_{n-1}, 1) \in k[t_0, \dots, t_{n-1}]$.

Now, by Lemma 1.4.1 the closure of $V(f) \cap U_n$ in $U_n = \mathbb{A}^n$ is cut out by the t_0 -saturation of the ideal (f') . Thus we must show this ideal is t_0 -saturated. Note that by construction,

$$f' = \bar{f}(t_0, t_1, \dots, t_{n-1}, 1) = f_d(t_1, \dots, t_{n-1}, 1) + t_0 \sum_{e < d} t_0^{d-e-1} f_e(t_1, \dots, t_{n-1}, 1)$$

is not divisible by t_0 . Let $g \in k[t_0, \dots, t_{n-1}]$ and suppose that f' divides $t_0^n g$. Since t_0 does not divide f' and the polynomial ring $k[t_0, \dots, t_{n-1}]$ is a UFD, we deduce that f' divides g . \square

2. Lecture 2 (Jan 15 and 27): Algebraic sets

Recommended reading: Kempf §1, Hartshorne I.3

2.1. Some general topology

Definition 2.1.1. Let X be a topological space.

- (a) We say that X is **irreducible** if for every pair of closed subsets $Y_1, Y_2 \subseteq X$ with $X = Y_1 \cup Y_2$, we have $X = Y_1$ or $X = Y_2$. (Equivalently, every non-empty open subset of X is dense.)
- (b) The **dimension** $\dim(X)$ is the supremum of the set of integers $n \geq 0$ for which there exists a chain

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of distinct closed irreducible subsets of X .

- (c) We say that X is **Noetherian** if every decreasing sequence $F_0 \supseteq F_1 \supseteq \dots$ of closed subsets stabilizes (is eventually constant).

Proposition 2.1.2. Let $X = V(f_1, \dots, f_r) \subseteq k^n$ be an affine algebraic set and let $A = k[T_1, \dots, T_n]/(f_1, \dots, f_r)$. Then:

- (a) X is irreducible if and only if the coordinate ring

$$\mathcal{O}(X) = k[T_1, \dots, T_n]/\mathcal{J}(X) = k[T_1, \dots, T_n]/\sqrt{(f_1, \dots, f_r)} = A/\sqrt{0}$$

is a domain, or equivalently iff the ideal $\sqrt{(f_1, \dots, f_r)} = \mathcal{J}(X) \subseteq k[T_1, \dots, T_n]$ is prime.

- (b) X is a Noetherian topological space.

- (c) We have $\dim(X) = \dim(\mathcal{O}(X)) = \dim(A)$ (the latter two denote the **Krull dimension**).

Remark 2.1.3. Thus irreducible closed subsets of X correspond to prime ideals of $\text{Spec}(A)$. The smallest irreducible closed subsets are the points of X , which correspond to the largest prime ideals, i.e. the maximal ideals.

Lemma 2.1.4. Let X be a Noetherian topological space. Then there exist closed irreducible subsets $Z_1, \dots, Z_r \subseteq X$ such that $Z_i \not\subseteq Z_j$ for $i \neq j$ and

$$X = Z_1 \cup \cdots \cup Z_r.$$

They are unique up to permutation.

Definition 2.1.5. The closed irreducible subsets $Z_1, \dots, Z_r \subseteq X$ in the lemma are called the **irreducible components** of X .

Note that we have $\dim(X) = \sup\{\dim Z_1, \dots, \dim Z_r\}$. In order to say more about dimension, we need to review some results from the **dimension theory** part of commutative algebra (see e.g. Chapter 10 of Atiyah–Macdonald, though we need a bit more).

Let $X \subseteq \mathbb{A}^n$ be an irreducible algebraic set, let $\mathfrak{p} = \mathcal{J}(X) \subseteq k[T_1, \dots, T_n]$ be the corresponding prime ideal, and let $A = k[T_1, \dots, T_n]/\mathfrak{p} = \mathcal{O}(X)$.

- (1) Let $K = \text{Frac}(A)$ be the field of fractions. Then

$$\dim(X) = \dim(A) = \text{trdeg}(K/k)$$

is the transcendence degree of the extension K/k . In particular, we have $\dim \mathbb{A}^n = n$.

(2) All maximal chains $Z_0 \subseteq \cdots \subsetneq Z_r$ of irreducible subsets of X have length $r = \dim(X)$.

From this we can deduce that for a locally closed $Y \subseteq \mathbb{A}^n$, we have $\dim(Y) = \dim(\overline{Y})$.

(3) Let $f \in A$ be a nonzero nonunit, and let $Y = V(f) \subseteq X$. Let $Y_1, \dots, Y_r \subseteq Y$ be the irreducible components of Y . Then

(a) We have $\dim(Y_i) = \dim(X) - 1$ for every $i = 1, \dots, r$.

(b) Conversely, A is a UFD (unique factorization domain, for example if $X = \mathbb{A}^n$), then every closed irreducible subset $Y \subseteq X$ such that $\dim(Y) = \dim(X) - 1$ is of the form $Y = V(f)$ for some prime element $f \in A$.

2.2. Regular functions

A polynomial $f \in k[T_1, \dots, T_n]$ defines a function $f: k^n \rightarrow k$. Similarly, a rational function $f = p/q \in k(T_1, \dots, T_n)$ with $p, q \in k[T_1, \dots, T_n]$, $q \neq 0$ defines a function $f = p/q: D(q) \rightarrow k$ where $D(q) = k^n \setminus V(q)$. Recall that the subsets $D(q)$ for a varying q form a basis of the Zariski topology on $k^n = \mathbb{A}^n$. We define regular functions as those which are *locally* given by a rational function.

Definition 2.2.1. Let $X \subseteq k^n$ be an algebraic set, $U \subseteq X$ an open subset, and let $f: U \rightarrow k$ be a function. We say that f is a **regular function** on U if every $x \in U$ there exists an open neighborhood $x \in V \subseteq U$ and $p, q \in k[T_1, \dots, T_n]$ such that for every $y \in V$, we have $q(y) \neq 0$ and

$$f(y) = \frac{p(y)}{q(y)}.$$

We observe first that if f is a regular function on U and $Z \subseteq U$ is a locally closed subset, then $f|_Z$ is a regular function. Moreover, every polynomial $f \in k[T_1, \dots, T_n]$ defines a regular function on \mathbb{A}^n and hence on every locally closed $U \subseteq \mathbb{A}^n$. For an affine algebraic set $X \subseteq \mathbb{A}^n$ we obtain a map

$$A = \mathcal{O}(X) = k[T_1, \dots, T_n]/\mathcal{J}(X) \longrightarrow \{\text{regular functions on } X\}.$$

It follows from the Nullstellensatz that this map is injective. Moreover, regular functions on any locally closed subset $U \subseteq \mathbb{A}^n$ form a k -subalgebra of the ring of all functions $U \rightarrow k$.

Theorem 2.2.2. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set and let $A = k[T_1, \dots, T_n]/\mathcal{J}(X)$ be its coordinate ring. Then:

(a) The map

$$A \longrightarrow \{\text{regular functions on } X\}$$

is an isomorphism of k -algebras.

(b) For $g \in A$, let $U = D(g) \subseteq X$. Then the above map induces an isomorphism

$$A[g^{-1}] \simeq \{\text{regular functions on } U\}.$$

The proof shows an algebraic variant of “partitions of unity” in differential geometry.

Proof. (a) We need to show that this map is surjective, so let f be a regular function on X and let $X = \bigcup V_i$ be an open cover such that $f|_{V_i} = p_i/q_i$ for $p_i, q_i \in k[T_1, \dots, T_n]$ with $V_i \subseteq D(q_i)$. We may assume that $V_i = D(g_i)$ for some $g_1, \dots, g_r \in A$ generating the unit ideal in A . We can simplify this a bit further: replacing g_i with $g_i q_i$ and p_i/q_i with $(p_i g_i)/(q_i g_i)$ we may assume that $g_i = q_i$. Consider the functions

$$f_i = q_i^2 f: X \rightarrow k.$$

We notice that $f_i = p_i q_i$ for every i (the right-hand side is the function $X \rightarrow k$ defined by the element $p_i q_i \in k[T_1, \dots, T_n]$). Indeed, on V_i we have $f_i = q_i^2 (p_i/q_i) = p_i q_i$, and outside of V_i both sides are zero.

Since $A = (q_1, \dots, q_r)$, we also have $A = (q_1^2, \dots, q_r^2)$. Let $a_1, \dots, a_r \in A$ be such that $1 = \sum a_i q_i^2$. Multiply the last equality by f to get

$$f = \sum_{i=1}^r a_i q_i^2 f = \sum_{i=1}^r a_i f_i = \sum_{i=1}^r a_i p_i q_i \in A.$$

(b) We apply (a) to $V(I, T_{n+1}g - 1) \subseteq \mathbb{A}^{n+1}$. □

2.3. Spaces with functions

Definition 2.3.1. Fix a field k . A **space with functions** (swf for short) is a topological space X together with an assignment, for every open $U \subseteq X$, of a k -subalgebra $\mathcal{O}(U)$ of the ring of all functions $U \rightarrow k$ (called the ring of **regular functions** on U) such that

- (a) Being regular is a local property. That is, if $U = \bigcup U_\alpha$ is an open cover and $f: U \rightarrow k$ is a function, then $f \in \mathcal{O}(U)$ if and only if $f|_{U_\alpha} \in \mathcal{O}(U_\alpha)$ for each α .
- (b) If $U \subseteq X$ is an open subset and $f \in \mathcal{O}(U)$, then the set

$$D(f) = \{x \in U : f(x) \neq 0\} \subseteq U$$

is an open subset of U and $(f|_{D(f)})^{-1} \in \mathcal{O}(D(f))$.

A **morphism** of spaces with functions is a continuous map $\phi: Y \rightarrow X$ such that the pullbacks of regular functions are regular: for every open $U \subseteq X$ and every regular function $f \in \mathcal{O}(U)$, the function $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$.

Note that a map of swf's $\phi: Y \rightarrow X$ induces a k -algebra homomorphism

$$\phi^*: \mathcal{O}(X) \longrightarrow \mathcal{O}(Y), \quad \phi^*(f) = f \circ \phi.$$

Remark 2.3.2 (If you know some sheaf theory). Condition (a) means that \mathcal{O} forms a *subsheaf* of the sheaf $\prod_{x \in X} k_x$ of all k -valued functions on X . Condition (b) ensures that the stalks $\mathcal{O}_x = \varinjlim_{U \ni x} \mathcal{O}(U)$ for $x \in X$ are local rings, with maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_x : f(x) = 0\}$.

Examples 2.3.3. (a) Let $k = \mathbb{R}$ or \mathbb{C} and let X be a topological space. Then $\mathcal{O}(U) = C(U; k)$ (continuous functions $U \rightarrow k$) gives X the structure of a space with functions.

(b) Similarly with C^∞ , analytic, and complex manifolds.

(c) If X is a space with functions and $U \subseteq X$ is an open subset, then U is a space with functions in the obvious way.

(d) Let again k be our chosen algebraically closed field and let $X \subseteq \mathbb{A}^n$ be a locally closed subset. For an open $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions on U as in Definition 2.3.1. This makes X into a space with functions. Note that by Theorem 2.2.2, the two meanings of $\mathcal{O}(X)$ we have introduced agree.

Theorem 2.3.4. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then for every space with functions Y , the pull-back map

$$\phi \mapsto \phi^*: \text{Hom}(Y, X) \longrightarrow \text{Hom}_k(\mathcal{O}(X), \mathcal{O}(Y))$$

is bijective.

Note that by Yoneda's lemma, this determines the swf X if we know the ring $\mathcal{O}(X)$. (There may be other swfs with the same $\mathcal{O}(X)$, but only one of them is an affine algebraic set.)

Proof. **Injectivity** is easy: if $\phi, \psi: Y \rightarrow X$ are two maps and $y \in Y$ is such that $\phi(y) = x \neq x' = \psi(y)$, we find an $f \in \mathcal{O}(X)$ with $f(x) \neq f(x')$ (for example, one of the coordinates T_1, \dots, T_n), and then $\phi^*(f)(y) = f(x) \neq f(x') = \psi^*(f)(y)$, and $\phi^* \neq \psi^*$.

Surjectivity: Let $\phi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ be a k -algebra homomorphism, for which we seek to build the corresponding map of swf's $\phi: Y \rightarrow X$. For each $y \in Y$, we have the evaluation map $\text{ev}_y: \mathcal{O}(Y) \rightarrow k$ mapping $f \mapsto f(y)$. Consider the composition

$$\mathcal{O}(X) \xrightarrow{\phi^*} \mathcal{O}(Y) \xrightarrow{\text{ev}_y} k.$$

This defines an element $x \in \text{Hom}_k(\mathcal{O}(X), k)$, which equals X by the Nullstellensatz. We define the map ϕ by $\phi(y) = x$. This defines a map (of sets) $\phi: Y \rightarrow X$ inducing ϕ^* . Moreover, the pull-back of the basic open set $D(f) \subseteq X$ is $D(\phi^*f) \subseteq Y$, which is open by axiom (b) of the definition of an swf, which shows that ϕ is continuous. We omit the (easy) verification that ϕ is a morphism of swf's. \square

Remark 2.3.5. Here is a direct way of reconstructing the swf X from the reduced k -algebra $A = \mathcal{O}(X)$. We set $X = \text{MSpec}(A) = \text{Hom}_k(A, k)$. We give it the induced topology from $\text{Spec}(A)$, in other words generated by the base open sets $D(g)$ for $g \in A$. Finally, we call a function $f: U \rightarrow k$ defined on an open $U \subseteq X$ regular for every $x \in U$ there exist $g, h \in A$ such that $D(g) \subseteq U$ and $f(y) = h(y)/g(y)$ for every $y \in D(g)$.

Corollary 2.3.6. *The category of affine algebraic sets (defined as a full subcategory of the category of swf's) is equivalent to the opposite category of the category of finitely generated reduced k -algebras.*

Corollary 2.3.7 (Products of affine algebraic sets). *The category of affine algebraic sets admits products. More precisely, let $X = V(I) \subseteq \mathbb{A}^n$ (with coordinates T_1, \dots, T_n) and $Y = V(J) \subseteq \mathbb{A}^m$ (with coordinates U_1, \dots, U_m) be two affine algebraic sets. Then*

$$X \times Y = V(I + J) \subseteq \mathbb{A}^{n+m}$$

is the product of X and Y in the category of swf's, and we have

$$\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

Proof. This is straightforward except for the fact that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. For this, see Proposition 5.17 in Milne's notes¹ (which also shows that $\mathcal{O}(X \times Y)$ is a domain if $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains. \square

We are finally able to define algebraic sets (which we will later identify as reduced schemes of finite type over k) and varieties.

Definition 2.3.8. Let k be an algebraically closed field.

- (1) An **algebraic set** over k is a space with functions X admitting a finite open cover $X = U_1 \cup \dots \cup U_n$ where each U_i is isomorphic as an swf to an affine algebraic set.
- (2) We say that an algebraic set is a **variety** if it is irreducible.
- (3) We say that X is **projective** if it is isomorphic to a projective algebraic set (see below).
- (4) We say that X is **quasi-affine** if it is isomorphic to an open subset of an affine algebraic set, and **quasi-projective** if it is isomorphic to an open subset of a projective algebraic set.

¹<https://www.jmilne.org/math/CourseNotes/AG.pdf>

We note the key fact that not only can every algebraic set be covered by open affine algebraic sets, but the **affine open subsets form a base** for the topology on X (since if $U \subseteq X$ is affine, then a basis of opens of U is given by the sets $D(g)$ for $g \in \mathcal{O}(U)$).

Example 2.3.9. The projective space \mathbb{P}^n has the standard open cover $U_0 \cup \dots \cup U_n$ by affine spaces. We call a function on a locally closed subset Z of \mathbb{P}^n regular if its restriction to each $Z \cap U_i$ is regular in the sense of Definition 2.3.1. Thus every locally closed subset of \mathbb{P}^n is an swf and moreover an algebraic set.

Example 2.3.10. The punctured plane $\mathbb{A}^2 \setminus \{0\}$ is quasi-affine but not affine (see below). Similarly, the punctured projective plane $\mathbb{P}^2 \setminus \{P\}$ for a point P is quasi-projective, but neither projective nor quasi-affine.

Lemma 2.3.11. (a) $\mathcal{O}(\mathbb{A}^{n+1} \setminus 0) = \mathcal{O}(\mathbb{A}^{n+1})$ for $n \geq 1$;

(b) $\mathcal{O}(\mathbb{P}^n) = k$;

(c) Let $X \subseteq \mathbb{P}^n$ be a projective variety (closed and irreducible subset). Then $\mathcal{O}(X) = k$.

Proof. (a) The set $U = \mathbb{A}^{n+1} \setminus 0$ is the union of $D(T_i)$, $i = 0, \dots, n$. We have $\mathcal{O}(D(T_i)) = \mathcal{O}(\mathbb{A}^{n+1})[T_i^{-1}]$. Consider all of these as subrings of $k[T_0^{\pm 1}, \dots, T_n^{\pm 1}]$, then $\mathcal{O}(U)$ is their intersection, which equals $k[T_0, \dots, T_n] = \mathcal{O}(\mathbb{A}^{n+1})$.

(b) For this we use the fact (easy proof omitted) that for an open (or locally closed) $W \subseteq \mathbb{P}^n$, a function $f: W \rightarrow k$ is regular if and only if $f \circ \pi$ is regular on $\pi^{-1}(W)$ where $\pi: U \rightarrow \mathbb{P}^n$ is the quotient map. The fact for $W = \mathbb{P}^n$ combined with (a) implies that $\mathcal{O}(\mathbb{P}^n)$ consists of all $f \in k[T_0, \dots, T_n]$ which are invariant under scaling of the coordinates, i.e. homogeneous of degree zero. But $k[T_0, \dots, T_n]_0 = k$.

(c) We shall prove this later. \square

Example 2.3.12 (Ojanguren). In all examples of algebraic sets we have encountered so far, the ring $\mathcal{O}(X)$ was a finitely generated k -algebra. This is true for affine algebraic sets and projective algebraic sets, but for completely different reasons. In general for an algebraic set X , the ring $\mathcal{O}(X)$ might be non-Noetherian. Here is a simple example, found by Ojanguren. Consider the projective three-space \mathbb{P}^3 with homogeneous coordinates $(X : Y : Z : T)$ and the subsets

$$W = V_{\mathbb{P}}(XY) \subseteq \mathbb{P}^3, \quad L = V_{\mathbb{P}}(X, Z) \subseteq W, \quad U = W \setminus L.$$

Thus W is the union of two hyperplanes $H_1 = V_{\mathbb{P}}(X), H_2 = V_{\mathbb{P}}(Y) \simeq \mathbb{P}^2$ in \mathbb{P}^3 intersecting along the line $V_{\mathbb{P}}(X, Y) \simeq \mathbb{P}^1$. The set $L \simeq \mathbb{P}^1$ is another line, contained in one of the planes H_1 and intersecting the other H_2 in a single point $Q = (0 : 0 : 0 : 1)$. Let us calculate $\mathcal{O}(U)$. A regular function f on U restricts to a regular function f_1 on $H_1 \setminus L \simeq \mathbb{A}^2$ (with coordinates $u = Y/Z$ and $v = T/Z$) and a regular function f_2 on $H_2 \setminus Z \simeq \mathbb{P}^2 \setminus Q$. But $\mathcal{O}(\mathbb{P}^2 \setminus Q) = k$, so f_2 is constant. It follows that

$$\mathcal{O}(U) = \{f \in k[u, v] : f(0, v) \in k\}$$

which is not Noetherian (as the ideal $v \cdot k[u, v]$ is contained in $\mathcal{O}(U)$ and is an ideal there, generated by vu^n for all $n \geq 0$ but not by any proper subset).

2.4. Problem session (Jan 27)

During the second problem session:

1. We discussed Ojanguren's example (Example 2.3.12).

2. As a prelude to products of (non-affine) algebraic sets and the **Segre embedding** (see Lecture 3), we constructed an isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric surface

$$Q = V(XY - ZW) \subseteq \mathbb{P}^2.$$

The map $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Q$ is given by

$$((u_0 : u_1), (v_0 : v_1)) \mapsto \left(\underbrace{u_0 v_0}_X : \underbrace{u_1 v_1}_Y : \underbrace{u_0 v_1}_Z : \underbrace{u_1 v_0}_W \right).$$

3. We introduced **algebraic groups**. An algebraic group is an algebraic set G endowed with a structure of a group for which the multiplication and inverse maps

$$\mu : G \times G \longrightarrow G, \quad \iota : G \rightarrow G$$

are morphisms of algebraic sets. We gave a list of examples:

- (a) The **additive group** \mathbb{G}_a , which is the affine line $\mathbb{A}^1 = k$ and the group structure is given by addition in k .
- (b) The **multiplicative group** \mathbb{G}_m , the punctured affine line $\mathbb{A}^1 \setminus 0 = k^\times$, where the group structure is given by multiplication.
- (c) The n -**roots of unity** $\mu_n \subseteq \mathbb{G}_m$, the subgroup of \mathbb{G}_m cut out by the equation $T^n - 1 = 0$. It is a finite group of order $n/\gcd(n, p^\infty)$ where $p = \text{char}(k)$ if the latter is positive and $p = 1$ otherwise.
- (d) The **general linear group** GL_n of invertible $n \times n$ matrices, which is the open subset $D(\Delta)$ of \mathbb{A}^{n^2} (with coordinates T_{ij}) given by the nonvanishing of the determinant $\Delta = \det[T_{ij}]$. The group structure is given by multiplication of matrices.
- (e) As we shall later learn **elliptic curve** (a smooth cubic in \mathbb{P}^2 with a chosen basepoint) has a unique group structure in which the basepoint is the neutral element. Moreover, this group structure is commutative.

Examples (a)–(d) are affine and (e) is not affine.

We discussed how **affine algebraic groups** correspond to commutative finite type and reduced **Hopf algebras** over k . Let G be an affine algebraic group and let $A = \mathcal{O}(G)$ be its coordinate ring, a finite type and reduced k -algebra. Then the multiplication μ and inverse ι correspond to k -algebra

$$\mu^* : A \rightarrow A \otimes_k A, \quad \iota^* : A \rightarrow A$$

(“comultiplication” and “coinverse”), satisfying the “duals” of the group axioms. For instance, associativity of G is expressed by the commutativity of the square

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \text{id}} & G \times G \\ \text{id} \times \mu \downarrow & & \downarrow \mu \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

which translates into the commutativity of

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\text{id} \otimes \mu^*} & A \otimes A \\ \mu^* \otimes \text{id} \uparrow & & \uparrow \mu^* \\ A \otimes A & \xleftarrow{\mu^*} & A. \end{array}$$

A k -algebra A endowed with a counit $\varepsilon^* : A \rightarrow k$, a comultiplication $\mu^* : A \rightarrow A \otimes A$ and a coinverse $\iota^* : A \rightarrow A$ satisfying these axioms is called a (commutative) **Hopf algebra** over k . The equivalence of categories between affine algebraic sets and the opposite category of reduced finite type k -algebras thus induces an equivalence between affine algebraic groups and the opposite category of reduced finite type Hopf algebras over k .

We computed that in the examples (a)–(d) above, the comultiplication maps are given by

- (a) $T \mapsto T_0 + T_1 : k[T] \rightarrow k[T_0, T_1]$
- (b) $T \mapsto T_0 T_1 : k[T, T^{-1}] \rightarrow k[T_0, T_0^{-1}, T_1, T_1^{-1}]$
- (c) $T \mapsto T_0 T_1 : k[T]/(T^n - 1) \rightarrow k[T_0, T_1]/(T_0^n - 1, T_1^n - 1)$
- (d) $T_{ij} \mapsto \sum_k T_{0,ik} T_{1,kj} : k[T_{ij}][\Delta^{-1}] \rightarrow k[T_{0,ij}, T_{1,ij}][\Delta_0^{-1}, \Delta_1^{-1}]$.

3. Lecture 3 (Jan 29): Products, separatedness, completeness

Recommended reading: Kempf §3

3.1. Basic facts about algebraic sets

In order to distinguish between regular functions on different spaces, we shall sometimes write $\mathcal{O}_X(U)$ instead of $\mathcal{O}(U)$ for the set of regular functions on an open $U \subseteq X$ of a space with functions X .

Definition 3.1.1. Let X be a space with functions and let $Y \subseteq X$ be a subspace (subset, endowed with the induced topology). The **induced swf structure** on Y is defined as follows: a function defined on an open of Y is regular if locally on U it extends to a regular function on an open of X .

To be completely precise: for an open $V \subseteq Y$ and $f: V \rightarrow k$, we have $f \in \mathcal{O}_Y(V)$ if and only if there exist opens $U_\alpha \subseteq X$ such that $V \subseteq \bigcup U_\alpha$ and regular functions $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that $f(y) = f_\alpha(y)$ for every $y \in V \cap U_\alpha$. One checks easily that Y endowed with \mathcal{O}_Y defined this way is an swf, and that the inclusion $Y \rightarrow X$ is a morphism of swf's.

Remark 3.1.2. A reader acquainted with sheaf theory will notice that \mathcal{O}_Y is the image of the morphism of sheaves $i^{-1}(\mathcal{O}_X) \rightarrow \prod_{y \in Y} k_y$. Here $i^{-1}(\mathcal{O}_X)$ is the sheaf pull-back of \mathcal{O}_X , and $\prod_{y \in Y} k_y$ is the sheaf of (not necessarily continuous) k -valued functions on Y .

In the context of affine algebraic sets, the way we have endowed $X = V(I) \subseteq \mathbb{A}^n$ with an swf structure shows that it is the induced swf structure from \mathbb{A}^n .

Definition 3.1.3. A morphism of swf's $Y \rightarrow X$ is an **immersion** if it is a homeomorphism onto its image and the swf structure on Y coincides with the induced swf structure on the image. A **closed** (resp. **open**, resp. **locally closed**) immersion is an immersion whose image is closed (resp. open, resp. locally closed) in X .

Example 3.1.4. Consider the map $f: \mathbb{A}^1 \rightarrow \mathbb{A}^2$ sending t to (t^2, t^3) . It is a homeomorphism onto its image, which is the cuspidal curve $C = V(Y^2 - X^3)$. However, it is not an immersion, since the map on coordinate rings is

$$f^*: k[X, Y]/(Y^2 - X^3) = \mathcal{O}(C) \longrightarrow \mathcal{O}(\mathbb{A}^1) = k[T], \quad f^*(X) = T^2, f^*(Y) = T^3$$

which is not an isomorphism.

Proposition 3.1.5. *Let X be an algebraic set and let $Y \subseteq X$ be a locally closed subset. Then Y is an algebraic set (when endowed with the induced swf structure).*

Proof. Let $X = U_1 \cup \dots \cup U_n$ be a finite affine open cover of X . Then $V_i = U_i \cap Y$ form an open cover of Y , and each V_i is locally closed in the affine algebraic set U_i . It follows that we may assume that X itself is affine (as an swf which is covered by a finite number of opens which are algebraic sets is an algebraic set).

Suppose first that Y is closed in X . Since X is affine, it is closed in \mathbb{A}^n . So Y is closed in \mathbb{A}^n and endowed with the induced swf structure, and hence an affine algebraic set.

Now, suppose that Y is open in X . In this case, since the standard open affines $D(f) \subseteq X$ (for $f \in \mathcal{O}(X)$) form a base of the topology on Y , we can write Y as the union of such opens. Moreover, since X is Noetherian, Y is quasi-compact, and hence a finite number suffices. \square

Let us record the following crucial fact which follows from the above proof:

Lemma 3.1.6. *Let X be an algebraic set. Then affine open subsets of X form a base of the topology on X .*

3.2. Products

Recall from the last lecture that if X and Y are *affine* algebraic sets, then $X \times Y$ is an affine algebraic set with coordinate ring

$$\mathcal{O}(X \times Y) \simeq \mathcal{O}(X) \otimes_k \mathcal{O}(Y).$$

A word of warning (see 1.3(d)): $X \times Y$ is the product of X and Y as sets, but not as topological spaces.

Theorem 3.2.1. *Let X and Y be algebraic sets. Then, the product $X \times Y$ of swf's exists and is an algebraic set. Moreover, if X and Y are projective (isomorphic to a closed algebraic subset of \mathbb{P}^n for some n), then so is $X \times Y$.*

Proof. The proof of the first part is straightforward (and boring). Cover $X = U_1 \cup \dots \cup U_n$ and $Y = V_1 \cup \dots \cup V_m$ with affine open subsets. Then, as sets

$$X \times Y = \bigcup_{i=1}^n \bigcup_{j=1}^m U_i \times V_j.$$

Each $U_i \times V_j$ has the structure of an affine algebraic set, with coordinate ring $\mathcal{O}(U_i) \otimes \mathcal{O}(V_j)$. We give $X \times Y$ the topology induced by the topologies on $U_i \times V_j$ (so $W \subseteq X \times Y$ is open iff $W \cap (U_i \times V_j)$ is open in $U_i \times V_j$ for all (i, j)), and deem a function $f: W \rightarrow k$ defined on an open $W \subseteq X \times Y$ regular if its restriction to $W \cap (U_i \times V_j)$ is regular for each pair (i, j) . We then need to verify that

- (a) Each $U_i \times V_j$ is an open subset of $X \times Y$.
- (b) If $W \subseteq X \times Y$ is an open contained in $U_i \times V_j$ for some (i, j) , then a function $f: W \rightarrow k$ is regular if and only if it is regular when treated as a function on an open subset of the affine algebraic set $U_i \times V_j$.
- (c) For any swf Z and maps $f: Z \rightarrow X, g: Z \rightarrow Y$, the resulting map $(f \times g): Z \rightarrow X \times Y$ is a morphism of swf's.

(The first two ensure that $X \times Y$ is an algebraic set, and the last one gives the universal property of the product.) I suggest you prove these statements, and look up the proof in a textbook in case you get stuck.

The assertion about projective algebraic sets is more fun to prove. Note first that it suffices to show that $\mathbb{P}^n \times \mathbb{P}^m$ is projective for every $n, m \geq 0$. Let X_0, \dots, X_n and Y_0, \dots, Y_m be their homogeneous coordinates. Set $N = (n+1)(m+1) - 1 = nm + n + m$ and consider $P = \mathbb{P}^N$ with homogeneous coordinates W_{ij} ($i = 0, \dots, n, j = 0, \dots, m$). Consider the map (called the **Segre embedding**)

$$\phi: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$$

defined by

$$((X_0 : \dots : X_n), (Y_0 : \dots : Y_m)) \mapsto (W_{ij} = X_i Y_j) = \begin{bmatrix} X_0 Y_0 & X_0 Y_1 & \dots & X_0 Y_m \\ X_1 Y_0 & \dots & & \dots \\ \dots & & \dots & \\ X_n Y_0 & \dots & & X_n Y_m \end{bmatrix}$$

Note first of all that this is a well-defined map of sets. Indeed, if we scale either all X_i 's or all Y_j 's by the same scalar $\lambda \in k^\times$, the result scales by the same factor, and if $X_i \neq 0$ and $Y_j \neq 0$, then $W_{ij} \neq 0$.

Next, we identify the image of this map. A matrix $[W_{ij}] \in \mathbb{P}^N$ is in the image if and only if it is of rank one (it cannot be of rank zero since not all coordinates vanish). On the other hand, this holds if and only if every 2×2 minor of this matrix is zero, and hence the image Q of ϕ is the projective algebraic set defined by the system of homogeneous equations

$$0 = \det \begin{bmatrix} W_{ij} & W_{i'j'} \\ W_{i'j} & W_{ij'} \end{bmatrix} = W_{ij} W_{i'j'} - W_{i'j} W_{ij'}, \quad i, i' \in \{0, \dots, n\}, \quad j, j' \in \{0, \dots, m\}$$

To check that the map ϕ induces an isomorphism onto \mathcal{Q} , we check what happens over the open subset $\mathbb{A}^N \simeq D(W_{00}) \subseteq \mathbb{P}^N$. Its preimage is defined by $X_0 \neq 0 \neq Y_0$ and hence it is equal to the open subset $D(X_0) \times D(Y_0) \simeq \mathbb{A}^n \times \mathbb{A}^m$ of $\mathbb{P}^n \times \mathbb{P}^m$. In the dehomogenized coordinates $w_{ij} = W_{ij}/W_{00}$, $x_i = X_i/X_0$, $y_j = Y_j/Y_0$, the intersection $\mathcal{Q} \cap D(W_{00})$ is cut out by the equations $w_{ij} = w_{0j}w_{i0}$ (set $(i', j') = (0, 0)$), and the restriction of ϕ is defined by the map of rings

$$k[w_{ij} : (i, j) \neq (0, 0)]/(w_{ij} - w_{0j}w_{i0}) \longrightarrow k[x_1, \dots, x_n, y_1, \dots, y_m], \quad w_{ij} \mapsto x_i y_j$$

(where we interpret $x_0 = 1$ and $y_0 = 1$) which is an isomorphism, the inverse sending x_i to w_{i0} and y_j to w_{0j} . \square

3.3. Separated varieties

Recall (or learn) the following fact from topology. A topological space is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) : x \in X\} \subseteq X \times X$$

is a closed subset of $X \times X$.

Definition 3.3.1. An algebraic set X is **separated** if the diagonal $\Delta \subseteq X \times X$ is a closed subset of $X \times X$.

Note that this does **not** imply that X is Hausdorff since the topology on $X \times X$ is not the product topology.

Examples 3.3.2. (a) If we have an injective map $f: Y \rightarrow X$ and X is separated, then so is Y . Indeed, then $\Delta_Y \subseteq Y \times Y$ is the preimage of the closed subset $\Delta_X \subseteq X \times X$ under the continuous map $f \times f: Y \times Y \rightarrow X \times X$.

(b) If X is (quasi)affine then X is separated. Indeed, then X admits an injective map to \mathbb{A}^n , and \mathbb{A}^n is separated, as its the diagonal in $\mathbb{A}^n \times \mathbb{A}^n$ (with coordinates $X_1, \dots, X_n, Y_1, \dots, Y_n$) is cut out by the equations $X_i = Y_i$ and hence is closed.

(c) The projective space \mathbb{P}^n is separated. (Consequently, every quasi-projective algebraic set is separated.) To see that \mathbb{P}^n is separated, we use the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^{n^2+2n}$. The diagonal is the preimage of the linear subvariety cut out by the equations $W_{ij} = W_{ji}$, and is therefore closed.

(d) (Line with doubled origin) Consider the space with functions X obtained by gluing two copies $U_i = \mathbb{A}^1$ (with coordinate T_i) for $i = 0, 1$ along the isomorphism of open subsets $D(T_0) \simeq D(T_1)$ sending T_1 to T_0 . There is a natural map $X \rightarrow \mathbb{A}^1$ (with coordinate T on the target pulling back to T_i on U_i) which is bijective away from zero, and such that $0 \in \mathbb{A}^1$ has two preimages $0_i \in U_i$. On $U_0 \times U_1$, the diagonal is $V(T_0 - T_1) \setminus \{(0_0, 0_1)\}$ and is not closed. Thus X is not separated.

Lemma 3.3.3. Let X be a separated algebraic set and let $U, V \subseteq X$ be affine open subsets. Then $U \cap V$ is affine.

Proof. We have $U \cap V \simeq (U \times V) \cap \Delta$, so $U \cap V$ is a closed subset of the affine algebraic set $U \times V$. \square

Example 3.3.4 (Plane with doubled origin). Consider a variant of Example 3.3.2(d) where we replace $U_i = \mathbb{A}^1$ with \mathbb{A}^2 with coordinates T_i, U_i and the open subsets $D(T_i)$ with $U_i \setminus \{(0, 0)\}$. The resulting space X has two affine opens U_0 and U_1 whose intersection $U_0 \cap U_1$ is isomorphic to $\mathbb{A}^2 \setminus \{(0, 0)\}$, which is not affine.

Again, recall from topology that two maps $f, g: Y \rightarrow X$ to a Hausdorff space X which are equal on a dense subset of Y have to be equal. Here is an algebraic variant:

Lemma 3.3.5. *Let $f, g: Y \rightarrow X$ be a parallel pair of maps between algebraic sets. If X is separated, then the subset (“equalizer”)*

$$\text{Eq}(f, g) = \{y \in Y : f(y) = g(y)\} \subseteq Y$$

is closed in Y .

Proof. Use the “diagonal trick:” $\text{Eq}(f, g) = (f \times g)^{-1}(\Delta_X)$ is the preimage of the diagonal $\Delta_X \subseteq X \times X$ under the map $f \times g: Y \rightarrow X \times X$. \square

3.4. Complete varieties

Recall (or learn) another fact from topology. A Hausdorff topological space X is compact if and only if for every topological space Y , the projection map

$$\pi_Y: X \times Y \longrightarrow Y$$

is closed (maps closed subsets of $X \times Y$ to closed subsets of Y).

Definition 3.4.1. An algebraic set X is **complete** (a.k.a. **proper**) if X is separated and for every algebraic set Y , the projection map

$$\pi_Y: X \times Y \longrightarrow Y$$

is closed.

Remark 3.4.2. (a) Again, $X \times Y$ does not have the product topology.

- (b) The affine space \mathbb{A}^n is not complete for $n \geq 1$. More generally, let X be an algebraic set admitting a function $f \in \mathcal{O}(X)$ which takes infinitely many values. Then X is not complete. Proof: consider $Y = \mathbb{A}^1$ with coordinate T and the closed subset $Z = V(fT - 1) \subseteq X \times Y$. Its projection onto Y is then an infinite subset which does not contain zero, and hence cannot be closed.
- (c) We shall prove later that \mathbb{P}^n is complete (and therefore, by Lemma 3.4.3, every projective algebraic set is complete).
- (d) The following relative versions of separatedness and completeness are used in algebraic geometry. A morphism $f: X \rightarrow S$ is **separated** if the diagonal $\Delta \subseteq X \times X$ is a closed subset of the fiber product

$$X \times_S X = \{(x, y) \in X \times X : f(x) = f(y)\} \subseteq X \times X.$$

A separated morphism $f: X \rightarrow S$ is **proper** if for every map of algebraic sets $g: Y \rightarrow S$, the projection map

$$\pi_Y: X \times_S Y \longrightarrow Y$$

is closed. Here $X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$.

- (e) A closed subspace of a complete algebraic set is complete.

Another standard fact from topology: if $f: Y \rightarrow X$ is a map from a compact space Y into a Hausdorff space X , then the image $f(Y) \subseteq X$ is a closed subspace of X (and is therefore both compact and Hausdorff). Algebraic version:

Lemma 3.4.3. *Let $f: Y \rightarrow X$ be a map from a complete algebraic set Y to a separated algebraic set X . Then $f(Y) \subseteq X$ is closed in X and complete.*

Proof. For the first statement, consider the graph

$$\Gamma_f = \{(y, x) : x = f(y)\} \subseteq Y \times X.$$

It is closed in $Y \times X$, being the equalizer of the projection $\pi_X : Y \times X \rightarrow X$ and the composition $f \circ \pi_Y : Y \times X \rightarrow X$ (here we use Lemma 3.3.5). Then $f(Y) = \pi_X(\Gamma_f)$ is closed in X since by assumption (Y complete) the map $X \times Y \rightarrow X$ is closed.

It remains to show that $Z = f(X)$ is complete. For this we use the surjective map $Y \rightarrow Z$ induced by f . Let W be an algebraic set and let $F \subseteq Z \times W$ be a closed subset. We must show that $\pi_W(F)$ is closed in W . Let $F' \subseteq Y \times W$ be the preimage of F . Since $Y \rightarrow Z$ is surjective, so is $F' \rightarrow F$, and hence $\pi_W(F) = \pi_W(F')$ is closed in W since $\pi_W : Y \times W \rightarrow W$ is closed. \square

Lemma 3.4.4. *Let X be a complete variety. Then $\mathcal{O}(X) = k$.*

Proof. Let $f \in \mathcal{O}(X)$, we must show that f is constant. Treat f as a morphism $f : X \rightarrow \mathbb{A}^1$. The image $f(X) \subseteq \mathbb{A}^1$ is then closed and complete by the previous lemma. It is also irreducible (being the image of the irreducible space X), and hence it is a singleton (because \mathbb{A}^1 is not complete, see Remark 3.4.2(a)). \square

Corollary 3.4.5. *Let X be an algebraic set which is both compact and quasi-affine. Then X is a finite set.*

4. Lecture 4 (Feb 3): Elimination theory

Recommended reading: Kempf §3

4.1. Statement of the result

Recall the following definitions from last time.

Definition 4.1.1. Let X be an algebraic set.

- (a) We say that X is **separated** if the diagonal $\Delta \subseteq X \times X$ is a closed subset of $X \times X$.
- (b) We say that X is **complete** if X is separated and for every algebraic set Y , the projection map

$$\pi_Y: X \times Y \longrightarrow Y$$

is closed.

Our goal today is to prove the following theorem.

Theorem 4.1.2. *The projective space \mathbb{P}^n is complete.*

We have shown in the previous lecture that \mathbb{P}^n is separated. Our goal is thus to show that for any algebraic set Y and any closed subset $Z \subseteq \mathbb{P}^n \times Y$, the image of Z in Y is closed.

Remark 4.1.3. Over $k = \mathbb{C}$, we can consider the “analytic topology” on an algebraic set X . Let us denote the resulting topological space by X^{an} . Then one can show that

$$X \text{ separated} \Leftrightarrow X^{\text{an}} \text{ Hausdorff}, \quad X \text{ complete} \Leftrightarrow X^{\text{an}} \text{ compact Hausdorff}.$$

In a course of topology or differential geometry, you might have seen that $\mathbb{C}P^n = (\mathbb{P}^n)^{\text{an}}$ is compact. Theorem 4.1.2 is thus an algebraic analog of this fact.

4.2. Warm-up: the resultant

Before tackling the proof of our theorem, let us deal with a special case (though at first it might not seem like a special case at all).

Let $f, g \in k[T]$ be two polynomials, $\deg(f) = n$, $\deg(g) = m$, $m, n > 0$. Write

$$f = \sum_{i=0}^n a_i T^i, \quad g = \sum_{i=0}^m b_i T^i.$$

The **resultant** of f, g is the determinant $R(f, g)$ of the $(n+m) \times (n+m)$ matrix

$$\begin{bmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & \vdots & b_1 & b_0 & \cdots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ a_n & \vdots & \ddots & a_0 & b_m & \vdots & \ddots & b_0 \\ 0 & a_n & \ddots & a_1 & 0 & b_m & \ddots & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_m \end{bmatrix} \tag{4.2.1}$$

Lemma 4.2.1. *Write $f = a_n \prod (T - \alpha_i)$ and $g = b_m \prod (T - \beta_j)$. We have*

$$R(f, g) = a_n^m b_m^n \prod_{i,j} (\alpha_i - \beta_j).$$

We won't need this, but only the following corollary, for which we supply an independent proof.

Corollary 4.2.2. *We have $R(f, g) = 0$ if and only if f and g have a common root.*

Proof. Since $k[T]$ is an PID, the polynomials f and g have no root in common if and only if they are coprime, i.e. if there exist polynomials $p, q \in k[T]$ such that

$$1 = pf + qg.$$

If we write $p = p_0g + p_1$ and $q = q_0f + q_1$ where $\deg(p_1) < m$ and $\deg(q_1) < n$, we have

$$1 = (p_0 + q_0)fg + p_1f + q_1g$$

Then $p_0 + q_0 = 0$, otherwise the right-hand side has degree $\geq n + m > 0$. Consequently, $1 = p_1f + q_1g$. In other words, we may assume that $\deg(p) < m$ and $\deg(q) < n$.

If we play the same game with the equation $h = pf + qg$ for $\deg(h) < n + m$, we obtain the following observation: For $i \geq 0$, let V_i be the space of polynomials of degree $< i$. Consider the linear map

$$\phi: V_m \oplus V_n \longrightarrow V_{n+m}, \quad \phi(p, q) = pf + qg.$$

Then f and g are coprime if and only if ϕ is surjective.

The result now follows since in the bases $(T^i, 0)$ ($i = 0, \dots, m-1$), $(0, T^i)$ ($i = 0, \dots, n-1$) in the source $V_m \oplus V_n$ and T^i ($i = 0, \dots, n+m-1$) in the target, the matrix of ϕ is (4.2.1). Thus ϕ is surjective if and only if

$$R(f, g) = \det(\phi) \neq 0. \quad \square$$

Remark 4.2.3. To deduce Lemma 4.2.1 from Corollary 4.2.2, fix the leading coefficients a_n and b_m , and treat the roots α_i, β_j as indeterminates (i.e., work over the polynomial ring $k[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m]$). The coefficients a_i ($i < n$) and b_i ($i < m$) are then expressed using standard symmetric polynomials in the α_i and β_j . Thus both sides of the equality in Lemma 4.2.1 are elements of this polynomial ring. By the corollary, we have $R(f, g) = 0$ if we substitute $\alpha_i = \beta_j$, which (by Nullstellensatz) implies that $(\alpha_i - \beta_j)$ divides $R(f, g)$. These linear polynomials are pairwise coprime, and hence the right-hand side divides the left-hand side. But the degrees and leading terms are the same (check by hand), so this is an equality. (N.B. The same strategy applies to the evaluation of the Vandermonde determinant.)

Since we are here, let us define the discriminant.

Definition 4.2.4. The **discriminant** of a polynomial $f = \sum_{i=0}^n a_i T^i \in k[T]$ of degree $n \geq 0$ is

$$\Delta(f) = (-1)^{n(n-1)/2} a_n^{-1} R(f, f')$$

where $f' = df/dT$ is the formal derivative of f .

Thus $\Delta(f) = 0$ if and only if f has a multiple root. For a quadratic $f = aT^2 + bT + c$, we have the familiar $\Delta(f) = b^2 - 4ac$, and for $f = T^3 + aT + b$ we have

$$\Delta(f) = -4a^3 - 27b^2$$

familiar from the theory of elliptic curves.

4.3. Proof of Theorem 4.1.2: Elimination theory

Theorem 4.3.1. *The projective space \mathbb{P}^n is complete.*

Proof. We have already shown that \mathbb{P}^n is separated (the diagonal being the preimage of the linear subspace $W_{ij} = W_{ij}$ under the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^N$, $N + 1 = (n + 1)^2$). It remains to show that for every algebraic set, the projection map $\mathbb{P}^n \times Y \rightarrow Y$ is closed.

Let Y be an algebraic set and let $Z \subseteq \mathbb{P}^n \times Y$ be a closed subset. Then $\pi_Y(Z) \subseteq Y$ is closed if and only if for every affine open $U \subseteq Y$, the subset $\pi_Y(Z) \cap U = \pi_U(Z \cap (\mathbb{P}^n \times U))$ is closed. It therefore suffices to treat the case $Y \subseteq \mathbb{A}^m$ closed. But then Z is closed in $\mathbb{P}^n \times \mathbb{A}^m$, and $\pi_Y(Z)$ is closed in Y if and only if it is closed in \mathbb{A}^n . We have now reduced to the case $Y = \mathbb{A}^n$.

Let T_0, \dots, T_n be the homogeneous coordinates on \mathbb{P}^n and let x_0, \dots, x_m be the coordinates on \mathbb{A}^n . Write $P = k[T_0, \dots, T_n] = \bigoplus_{d \geq 0} P_d$ and $B = k[x_0, \dots, x_m]$. Consider the graded polynomial ring

$$A = B \otimes_k P = k[x_0, \dots, x_m][T_0, \dots, T_n], \quad A_d = B \otimes_k P_d.$$

A homogeneous ideal $I \subseteq A$ defines a k^\times -invariant closed subset of $(\mathbb{A}^{n+1} \setminus 0) \times \mathbb{A}^m$ and hence a closed subset of $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$. Every closed subset of $\mathbb{P}^n \times \mathbb{A}^m$ is of this form (easy proof omitted).

Let thus $I \subseteq A$ be the radical homogeneous ideal corresponding to our closed subset $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$, and write $I = (f_1, \dots, f_r)$ where $f_i \in A_{d_i} = B \otimes_k P_{d_i}$. Then the image $\pi_{\mathbb{A}^n}(Z)$ is the set of all $(x_1, \dots, x_m) \in k^n$ for which the system

$$f_i(x_1, \dots, x_m, T_0, \dots, T_n) = 0, \quad i = 1, \dots, r$$

has a nonzero solution $(t_0, \dots, t_n) \in k^{n+1}$. Effectively, we wish to eliminate the variables T_0, \dots, T_n from this system. For this we need:

Claim. *Let $f_1, \dots, f_r \in P = k[T_0, \dots, T_n]$ be homogeneous, $f_i \in P_{d_i}$. Then the system $f_i = 0$ has no nonzero solution in k^{n+1} if and only if for some $d \geq 0$, every $f \in P_d$ can be written as*

$$f = \sum_{i=1}^r h_i f_i, \quad h_i \in P_{d-d_i}.$$

(In other words, if $I_{d'} = P_{d'}$ for some $d' \geq 0$. If this holds for d' , then it also holds for all $d'' > d'$, so we can rephrase the condition as: $I_d = P_d$ for $d \gg 0$.)

The claim is almost obvious: having no nonzero solutions means that $V(f_1, \dots, f_r) \subseteq \{0\} = V(T_0, \dots, T_n)$. Applying $\mathcal{J}(-)$ translates this to $I = (T_0, \dots, T_n) \subseteq \sqrt{(f_1, \dots, f_r)}$, i.e. $T_i^N \in I$ for large enough N . But this means that $I_d = P_d$ for $d \gg 0$ (more precisely, $d > N(n+1)$ will do).

Let us rephrase the condition from the claim: the system $f_i = 0$ has a nonzero solution if and only if for every $d \geq 0$, the map

$$\phi_d: \bigoplus_{i=1}^r P_{d-d_i} \longrightarrow P_d, \quad \phi_d(h_1, \dots, h_r) = \sum_{i=1}^r h_i f_i$$

is not surjective. Note that this is a map between finite-dimensional vector spaces over k , corresponding to a big rectangular matrix, say of size $a_d \times b_d$ (the exact values of $a_d = \sum \dim(P_{d_i})$ and $b_d = \dim(P_d)$ are unimportant). Its non-surjectivity can thus be detected by the vanishing of all minors of size $b_d \times b_d$.

Now come back to our initial problem: our f_1, \dots, f_n depend on the parameters x_1, \dots, x_m . We consider the map between free modules of finite rank (a_d and b_d) over the polynomial ring $B = k[x_1, \dots, x_m]$:

$$\phi_d: \bigoplus_{i=1}^r B \otimes_k P_{d-d_i} \longrightarrow B \otimes_k P_d, \quad \phi_d(h_1, \dots, h_r) = \sum_{i=1}^r h_i f_i.$$

By the claim, the image $\pi_{\mathbb{A}^n}(Z)$ is the set of points (x_1, \dots, x_m) at which ϕ_d . By the previous discussion, it is cut out by the ideal generated by the $b_d \times b_d$ minors of the corresponding matrix (now, treated as elements of $k[x_1, \dots, x_m]$) and is therefore closed. \square

Corollary 4.3.2. *Every projective algebraic set is complete.*

Remark 4.3.3. Consider the case $n = 1$ and $Z \subseteq \mathbb{P}^1 \times Y$ cut out by a pair of functions $f = g = 0$ where

$$f = \sum_{i=0}^n a_i T_0^{n-i} T_1^i, \quad g = \sum_{i=0}^m b_i T_0^{m-i} T_1^i$$

for $a_0, \dots, a_n, b_1, \dots, b_m \in \mathcal{O}(Y)$ and $a_n, b_m \in \mathcal{O}(Y)^\times$. Then the proofs of Corollary 4.2.2 and of Theorem 4.1.2 give the same description of the image of Z in Y as the vanishing set $V(R)$ of the resultant $R = R(f(1, T), g(1, T)) \in \mathcal{O}(Y)$ (we substituted $T_0 = 1$ to de-homogenize the polynomials).

4.4. Chevalley's theorem

What can we say about the image of a morphism $Y \rightarrow X$ between algebraic sets? If Y is complete and X is separated, then the image is closed. In general, the image is a constructible subset.

Definition 4.4.1. Let X be an algebraic set. A subset $W \subseteq X$ is **constructible** if it is the union of a finite number of locally closed subsets of X .

Importantly, constructible subsets of X form a Boolean algebra (closed under intersection, union, and complement).

Remark 4.4.2. A word of warning: While every locally closed subset of an algebraic set is an algebraic set, not every constructible subset is an algebraic set. For example, the subset

$$\{(0, 0)\} \cup D(X) \subseteq \mathbb{A}^2$$

(where the coordinates are X, Y) is constructible but not locally closed, and it does not have any obvious structure of an algebraic set.

Last semester, we proved the following result:

Theorem 4.4.3 (Chevalley, algebraic version). *If $A \rightarrow B$ is a morphism of finite type between Noetherian rings, then the image of*

$$\text{Spec}(B) \longrightarrow \text{Spec}(A)$$

is constructible.

We can now deduce the geometric counterpart.

Theorem 4.4.4 (Chevalley, geometric version). *Let $f: Y \rightarrow X$ be a morphism between algebraic sets and let $W \subseteq Y$ be a constructible subset. Then $f(W)$ is a constructible subset of X .*

Proof. Theorem 4.4.3 implies the result if X and Y are affine and $W = Y$. The general case is deduced from this by passing to affine open covers (details omitted). \square

5. Lecture 5 (Feb 5): Local rings, rational maps

Recommended reading: Hartshorne I.4

5.1. Local rings, function fields, and rational maps

Let X be an affine algebraic set with coordinate ring $A = \mathcal{O}(X)$. Recall that irreducible closed subsets $Z \subseteq X$ correspond to prime ideals $\mathfrak{p} \subseteq A$. If Z is an irreducible subset, then every non-empty open subset is dense, and the intersection of two non-empty opens is non-empty.

Definition 5.1.1. Let X be a (not necessarily affine) algebraic set and let $Z \subseteq X$ be an irreducible closed subset. We define the **stalk at Z** (also called the **local ring at Z**) as the filtered colimit

$$\mathcal{O}_{X,Z} = \varinjlim_{U \cap Z \neq \emptyset} \mathcal{O}(U)$$

over all open subsets $U \subseteq X$ which intersect Z (this is a filtered colimit since Z is irreducible, by the previous remark).

In plain terms, by the basic properties of filtered colimits, an element of $\mathcal{O}_{X,Z}$ is an equivalence class of pairs (U, f) where $U \subseteq X$ is an open intersecting Z and where $f \in \mathcal{O}(U)$, where we identify (U, f) with (U', f') if there exists an open $U'' \subseteq U \cap U'$ intersecting Z such that $f|_{U''} = f'|_{U''}$.

Examples 5.1.2. (a) Let $x \in X$. Then $\{x\} \subseteq X$ is closed and irreducible, and we denote the ring $\mathcal{O}_{X,\{x\}}$ more simply by $\mathcal{O}_{X,x}$. Its elements are **germs** of regular functions defined in a neighborhood of x .

(b) At the other extreme, suppose that X is itself irreducible (i.e. a “variety”). In this case we can take $Z = X$. We denote the ring $\mathcal{O}_{X,X}$ more simply by $k(X)$ and call it the **function field** of X . Its elements are represented by regular functions defined on a non-empty open subset of X .

Remark 5.1.3 (From the future). We shall later notice that $\mathcal{O}_{X,x}$ is simply the *stalk* of the structure sheaf \mathcal{O}_X (the assignment $U \mapsto \mathcal{O}(U)$). Moreover, scheme theory extends this interpretation to $\mathcal{O}_{X,Z}$ for an arbitrary closed irreducible Z . Namely, the scheme X^{sch} corresponding to X adds a unique generic point η_Z for every closed irreducible $Z \subseteq X$. Then $\mathcal{O}_{X,Z} \simeq \mathcal{O}_{X^{\text{sch}},\eta_Z}$.

Lemma 5.1.4. Let X be an algebraic set and let $Z \subseteq X$ be an irreducible closed subset.

(a) Let $U \subseteq X$ be an affine open intersecting Z , so that $Z \cap U$ is an irreducible closed subset of U . Let $A = \mathcal{O}(U)$ and let $\mathfrak{p} = \mathcal{J}(Z \cap U) \subseteq A$ be the prime ideal corresponding to $Z \cap U$. Then

$$\mathcal{O}_{X,Z} = \mathcal{O}_{U,U \cap Z} = A_{\mathfrak{p}}.$$

(Recall that $A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$.) In particular, $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal consisting of germs of functions (U, f) which vanish on $Z \cap U$.

(b) If X is irreducible, then $k(X)$ is a field, equal to the fraction field of $\mathcal{O}(U)$ (which is a domain) for every non-empty affine open $U \subseteq X$.

(c) The residue field of the local ring $\mathcal{O}_{X,Z}$ is the function field $k(Z)$ of Z .

Proof. (a) We first notice that if $U \subseteq X$ is any open intersecting Z , then $\mathcal{O}_{X,Z} \simeq \mathcal{O}_{U,U \cap Z}$. This follows from abstract properties of filtered colimits (those $V \subseteq X$ which are contained in U are cofinal among those which intersect Z), and is also easy to show directly. Namely, if (V, f) is an element of $\mathcal{O}_{X,Z}$, we get a corresponding element $(V \cap U, f|_{V \cap U})$ of $\mathcal{O}_{U,U \cap Z}$, and if (V, f) is an element of $\mathcal{O}_{U,U \cap Z}$, then (V, f) also represents an element of $\mathcal{O}_{X,Z}$. We verify immediately that these correspondences give well-defined mutually inverse ring homomorphisms.

To obtain the rest, we may assume that $X = U$ is affine. Recall that in this case X has a base of the topology consisting of standard opens $D(f)$ for $f \in A$. Therefore, if $V \subseteq X$ is an open intersecting Z , then there exists an $f \in A$ such that $D(f) \subseteq V$ and $D(f) \cap Z \neq \emptyset$. In defining $\mathcal{O}_{X,Z}$, we can thus restrict to a filtered colimit over opens only of this special form. Notice that the last condition $D(f) \cap Z \neq \emptyset$ is equivalent to $f \notin \mathfrak{p}$. Moreover, we have proved that $\mathcal{O}(D(f)) = A[f^{-1}]$. Assembling these observations, we get

$$\mathcal{O}_{X,Z} = \varinjlim_{D(f) \cap Z \neq \emptyset} \mathcal{O}(D(f)) = \varinjlim_{D(f) \cap Z \neq \emptyset} A[f^{-1}] = A_{\mathfrak{p}}.$$

As shown in commutative algebra, this is a local ring with maximal ideal $\mathfrak{p} \cdot A_{\mathfrak{p}}$, which coincides with those $f \in \mathcal{O}_{X,Z}$ which vanish on Z .

(b) Apply (a) to $X = Z$, in which case $\mathfrak{p} = (0)$ and $A_{\mathfrak{p}}$ is the fraction field of A .

(c) Let $U \subseteq X$ be an affine open intersecting Z and let $A = \mathcal{O}(U)$. Recall that the residue field $A_{\mathfrak{p}}/\mathfrak{p} \cdot A_{\mathfrak{p}}$ of $\mathcal{O}_{X,Z} = A_{\mathfrak{p}}$ (denoted by $\kappa(\mathfrak{p})$ in commutative algebra) can also be expressed as the fraction field of A/\mathfrak{p} . Now $A/\mathfrak{p} = \mathcal{O}(U \cap Z)$, so the result follows from (b) applied to $U \cap Z$. \square

Remark 5.1.5 (Direct proof of a part of (a) for swf's). The fact that $\mathcal{O}_{X,Z}$ is a local ring with maximal ideal consisting of functions vanishing on Z holds more generally for any swf X and a closed irreducible $Z \subseteq X$. We give the direct argument here since it illustrates well the motivation behind the second axiom of an swf.

Let $\mathfrak{m} \subseteq \mathcal{O}_{X,Z}$ be the set of all $(U, f) \in \mathcal{O}_{X,Z}$ such that $f|_{U \cap Z} = 0$. We verify immediately that (1) this condition does not depend on the choice of representative (this uses Z irreducible), and (2) is an ideal of $\mathcal{O}_{X,Z}$.

In order to check that $\mathcal{O}_{X,Z}$ is local with maximal ideal \mathfrak{m} , we need to show that every (U, f) which does not belong to \mathfrak{m} is an invertible element of $\mathcal{O}_{X,Z}$. The condition that $(U, f) \notin \mathfrak{m}$ means precisely that $D(f)$ meets Z . By axiom (2) of the definition of a space with functions, this set is open and $1/f \in \mathcal{O}(D(f))$. Thus $(U, f) \cdot (D(f), 1/f) = 1$ in $\mathcal{O}_{X,Z}$, and (U, f) is invertible.

Lemma 5.1.6. *Let $\phi: Y \rightarrow X$ be a morphism between algebraic sets and let $W \subseteq Y$ be an irreducible closed subset. Let $Z \subseteq X$ be the closure of $\phi(Z)$. Then Z is also irreducible, and ϕ induces a local homomorphism of local rings*

$$\phi^*: \mathcal{O}_{X,Z} \longrightarrow \mathcal{O}_{Y,W}.$$

Proof. That $\phi(W)$ and its closure Z are both irreducible is easy general topology. Moreover, $\phi(W)$ is dense in Z by definition. Therefore, if $U \subseteq X$ is an open intersecting Z , it has to intersect $\phi(W)$, and thus $\phi^{-1}(U) \subseteq Y$ is an open intersecting W . The pull-back maps $\phi^*: \mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(\phi^{-1}(U))$ for varying U induce map on filtered colimits

$$\phi^*: \mathcal{O}_{X,Z} = \varinjlim_U \mathcal{O}_X(U) \longrightarrow \varinjlim_U \mathcal{O}_Y(\phi^{-1}(U))$$

(both colimits over opens $U \subseteq X$ meeting Z) which we compose with the natural map (from the universal property of direct limit!)

$$\varinjlim_U \mathcal{O}_Y(\phi^{-1}(U)) \longrightarrow \varinjlim_V \mathcal{O}_Y(V) = \mathcal{O}_{Y,W}$$

(second colimit over opens $V \subseteq Y$ meeting W) to obtain the desired map $\phi^*: \mathcal{O}_{X,Z} \rightarrow \mathcal{O}_{Y,W}$. This homomorphism is local thanks to Lemma 5.1.4(a): if $f \in \mathcal{O}_X(U)$ vanishes on $U \cap Z$ then $\phi^*(f) = f \circ \phi \in \mathcal{O}_Y(\phi^{-1}(U))$ vanishes on $\phi^{-1}(U \cap Z) \supseteq \phi^{-1}(U) \cap W$. \square

We shall now consider rational maps between varieties. For this, let us note the following straightforward corollary of Lemma 5.1.6. For this, let us call a map $\phi: Y \rightarrow X$ **dominant** if $\phi(Y)$ is dense in X (and hence, by Chevalley's theorem, contains a dense open subset of X).

Corollary 5.1.7. *A dominant map between varieties $\phi: Y \rightarrow X$ induces an extension of function fields $\phi^*: k(X) \hookrightarrow k(Y)$.*

Proof. Apply Lemma 5.1.6 to $W = Y$, so $\mathcal{O}_{Y,W} = k(Y)$. Since ϕ is dominant, $\phi(W) = \phi(Y)$ is dense, and we have $Z = X$, so $\mathcal{O}_{X,Z} = k(X)$. \square

A rational map is a germ of a function between varieties.

Definition 5.1.8. Let X and Y be varieties (i.e. irreducible algebraic sets). A **rational map** from Y to X is an equivalence class of pairs (U, f) where $U \subseteq Y$ is a non-empty open subset and where $f: U \rightarrow X$ is a map of varieties, where we identify (U, f) and (U', f') if $f = f'$ on some non-empty open $U'' \subseteq U \cap U'$. We call a rational map (U, f) **dominant** if $f(U)$ is dense in X (this condition depends only on the equivalence class of (U, f)).

Remark 5.1.9. (a) A dominant rational map (U, f) from Y to X induces a pull-back map $f^*: k(X) \rightarrow k(Y)$ between the function fields.

(b) Conversely, let $k(X) \rightarrow k(Y)$ be a map of k -algebras. Then there exists a unique dominant rational map $Y \rightarrow X$ inducing this field extension.

(c) Dominant rational maps can be composed. The resulting category of varieties and rational maps is equivalent to the opposite of the category of finitely generated field extensions of k .

(d) If $V \subseteq Y$ is an open such that a given rational map from Y to X is represented by a pair (V, f) , we say that f is **defined on V** . If Y is separated, there exists a largest open $V \subseteq Y$ on which a given rational map is defined.

Definition 5.1.10. A rational map f from Y to X is **birational** if it is dominant and if it admits an inverse (in the category of dominant rational maps), or equivalently if there exist non-empty opens $V \subseteq Y$ and $U \subseteq X$ such that f induces an isomorphism $V \simeq U$, or equivalently if it induces an isomorphism $k(X) \simeq k(Y)$. We say that two varieties X and Y are **birational** if there exists a birational rational map from Y to X . We say that a variety X is **rational** if it is birational to \mathbb{P}^n for some $n \geq 0$.

Example 5.1.11. We shall prove later that the cubic plane curve $V(Y^2 - X^3 - X)$ is not rational.

6. Lecture 6 (Feb 10): Nonsingular varieties

Recommended reading: Hartshorne I.4, I.5, I.6

6.1. More on birational equivalence

We complement our discussion of birational equivalences from last time.

Lemma 6.1.1 (Hartshorne Corollary I 4.5). *Let X and Y be varieties. The following are equivalent:*

- (a) *There exist dominant rational maps $\phi: Y \dashrightarrow X$ and $\psi: Y \dashrightarrow X$ such that $\phi \circ \psi = \text{id}_X$ and $\psi \circ \phi = \text{id}_Y$;*
- (b) *$k(X) \simeq k(Y)$ as extensions of k ;*
- (c) *there exist non-empty opens $U \subseteq X$ and $V \subseteq Y$ such that $U \simeq V$ as varieties.*

Proof. (a) \Leftrightarrow (b): Proved earlier.

(a) \Rightarrow (c): Represent ϕ by a map $\phi: V' \rightarrow X$ and ψ by a map $\psi: U' \rightarrow Y$. Set $U = \psi^{-1}(\phi^{-1}(U'))$ and $V = \phi^{-1}(\psi^{-1}(V'))$. Then ϕ and ψ restrict to inverse isomorphisms $U \simeq V$.

(c) \Rightarrow (a): obvious. □

The following algebraic fact is a bit complicated to show, but not too difficult.

Theorem 6.1.2 (Zariski–Samuel vol. I, Chapter II, Theorem 31, p. 105). *Let k be a perfect field and let K be a finitely generated field extension of k . Then K is **separably generated** over k , i.e. it can be written as a finite separable² extension of $k(T_1, \dots, T_n)$ where $n = \text{trdeg}(K/k)$.*

Recall from CA that every finite separable extension is generated by a single element (the “primitive element theorem”). More precisely, in the situation in the theorem, we can write $K = k(T_1, \dots, T_n)[T]/(f)$ where $f \in k(T_1, \dots, T_n)[T]$ is an irreducible separable polynomial (that is, $f' \neq 0$).

Corollary 6.1.3. *Every variety is birational to an irreducible hypersurface in \mathbb{P}^{n+1} .*

Proof. Let X be a variety. Apply Theorem 6.1.2 to $K = k(X)$, writing $k(X)$ as a finite separable extension of $k(T_1, \dots, T_n)$, $n = \text{trdeg}(k(X)/k) = \dim X$. By the primitive element theorem (see discussion above), we can write

$$k(X) = k(T_1, \dots, T_n)[T]/(f)$$

for a separable irreducible polynomial $f \in k(T_1, \dots, T_n)[T]$. Clearing denominators, we obtain an irreducible polynomial $f \in k[T_1, \dots, T_n, T]$. Let $g \in k[T_0, T_1, \dots, T_n, T]$ be its homogenization (which is again irreducible) and let $Y = V_{\mathbb{P}}(g) \subseteq \mathbb{P}^{n+1}$. Then $k(Y) \simeq k(X)$ by construction, and so X and Y are birational. □

6.2. Nonsingular varieties

Recall that a Noetherian local ring A is **regular** if

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A).$$

Here $\mathfrak{m} \subseteq A$ is the unique maximal ideal and $k = A/\mathfrak{m}$ is its residue field. In general, we have \geq instead of equality, and the left-hand side coincides with the minimal number of generators of \mathfrak{m} . Every regular ring is a UFD (this is not so easy to show).

²See the note on separable extensions from last semester, available at <https://achinger.impan.pl/ca2025/sep.pdf>

Definition 6.2.1. Let X be an algebraic set and let $x \in X$. We say that X is **nonsingular** at x if $\mathcal{O}_{X,x}$ is a regular local ring.

We shall also call x a **smooth** or **regular** point. If every point is nonsingular, we say that X itself is nonsingular/smooth/regular.

Examples 6.2.2. (a) If $\dim(X) \leq 1$, the ring $\mathcal{O}_{X,x}$ is regular if and only if it is a discrete valuation ring (or equal to k in case of isolated points), if and only if it is integrally closed.

(b) The affine space \mathbb{A}^n is nonsingular. Indeed, for every $x = (x_1, \dots, x_n) \in \mathbb{A}^n$ we have $\dim(\mathcal{O}_{X,x}) = n$ and the maximal ideal \mathfrak{m} is generated by n elements $T_i - x_i$.

(c) Let $X = V(f) \subseteq \mathbb{A}^n$ be a hypersurface (where $f \neq 0$) and let $x \in X$. Then X is nonsingular at x if and only if $(\partial f / \partial T_i)(x) \neq 0$ for some i . Proof: Since $\dim(\mathcal{O}_{X,x}) = n - 1$, we want $\dim(\mathfrak{m}/\mathfrak{m}^2) = n - 1$. If $\mathfrak{n} = (T_1 - x_1, \dots, T_n - x_n) \subseteq k[T_1, \dots, T_n]$ is the ideal corresponding to x , then

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{n}/((\mathfrak{n}^2 + (f)))$$

which has dimension either n or $n - 1$, and the latter precisely when $f \notin \mathfrak{n}^2$. We can write

$$f(T_1, \dots, T_n) = \sum_{i=1}^n \frac{\partial f}{\partial T_i}(x) \cdot (T_i - x) + R$$

where $R \in \mathfrak{n}^2$. Thus $f \in \mathfrak{n}^2$ precisely when $(\partial f / \partial T_i)(x) = 0$ for all i .

Theorem 6.2.3. Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $I = \mathcal{J}(X) = (f_1, \dots, f_r)$ be its ideal. Then a point $x \in X$ is nonsingular if and only if

$$\text{rank} \left[\frac{\partial f_j}{\partial T_i}(x) \right] = n - \dim(X).$$

Proof. We did the case $r = 1$. For the general case see Theorem I 5.1 in Hartshorne. □

In general, for two nonsingular points $x \in X$ and $y \in Y$, the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are often non-isomorphic even if they have the same dimension. Indeed, we have $k(X) = \text{Frac}(\mathcal{O}_{X,x})$, so $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}$ only if X and Y are birational. So our intuition from differential geometry that a smooth variety should locally look like \mathbb{A}^n , taken too literally, is false. This is because $\mathcal{O}_{X,x}$ is defined in terms of Zariski open neighborhoods of x , which are very large. One can resolve this issue either by considering the étale topology (which we might cover later) or by completing the local ring.

Theorem 6.2.4 (Special case of Cohen's structure theorem). Let X be an algebraic set and let $x \in X$. Denote by

$$\widehat{\mathcal{O}}_{X,x} = \varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^n$$

the **completion** of the local ring $\mathcal{O}_{X,x}$ with respect to its maximal ideal. Then x is a nonsingular point of X if and only if

$$\widehat{\mathcal{O}}_{X,x} \simeq k[[T_1, \dots, T_n]].$$

More precisely, if $f_1, \dots, f_n \in \mathfrak{m}_x$, we have a unique continuous homomorphism

$$\theta: k[[T_1, \dots, T_n]] \longrightarrow \widehat{\mathcal{O}}_{X,x}, \quad \theta(T_i) = f_i.$$

The map θ is surjective if and only if $\mathfrak{m}_x = (f_1, \dots, f_n)$, and an isomorphism if and only if in addition

$$n = \dim(k[[T_1, \dots, T_n]]) = \dim(\mathcal{O}_{X,x}),$$

which happens precisely if $\mathcal{O}_{X,x}$ is regular.

Theorem 6.2.5. *Let X be a variety. Then the set $\text{Sm}(X)$ of nonsingular points of x is a non-empty open subset of X .*

Proof. Theorem 6.2.3 implies that $\text{Sm}(X)$ is open. We need to show it is non-empty. For this, we are going to follow [Hartshorne, Theorem I 3.5] which uses Theorem 6.1.2 above. See [Kempf, Lemma 6.1.6] for a fun and elementary proof of the same fact.

By Corollary 6.1.3 we may assume that $X = V(f) \subseteq \mathbb{A}^{n+1}$ is an affine hypersurface. The construction ensures that the projection $X \rightarrow \mathbb{A}^n$ onto the first n coordinates is “separable,” more precisely that $(f, f') = 1$ in $k(T_1, \dots, T_n)[T]$ where $f' = df/dT$. This means that we can write $1 = \alpha f + \beta f'$ in $k(T_1, \dots, T_n)[T]$. Take $t = (t_1, \dots, t_n) \in \mathbb{A}^n$ where the denominators of α and β do not vanish and a point $x \in X$ above t . Then $\alpha f + \beta f' = 1$ implies that $f'(x) \neq 0$, and so X is smooth at x . \square

Remark 6.2.6 (Resolution of singularities). One of the most important tools in algebraic geometry is the ability to reduce questions about arbitrary varieties to the case of nonsingular varieties by means of “resolution of singularities,” established in 1970 by Hironaka. Hironaka’s theorem says the following: let X be a projective variety and assume that k has **characteristic zero**, then there exists a birational map $\pi: X' \rightarrow X$ such that X' is a nonsingular projective variety. We will learn how to do this for curves (by means of “normalization”). Already for surfaces, the problem becomes nontrivial (but was resolved by Zariski in the 1940s). Nobody knows how to resolve singularities in positive characteristic beyond dimension 3, despite continued effort.

6.3. Nonsingular curves

By a **curve** we shall mean a separated one-dimensional variety.

Lemma 6.3.1. *Let C be a curve and let $x \in C$. The following are equivalent:*

- (a) C is non-singular at x i.e. $\mathcal{O}_{C,x}$ is a regular local ring;
- (b) $\mathcal{O}_{C,x}$ is a discrete valuation ring;
- (c) $\mathcal{O}_{C,x}$ is integrally closed;
- (d) the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{C,x}$ is principal.

Proof. See Atiyah–Macdonald, Proposition 9.2. \square

Remark 6.3.2. The algebra behind this proof can be used to show more generally that if X is a variety of dimension d which is normal (i.e. $\mathcal{O}(U)$ is integrally closed for every non-empty affine open $U \subseteq X$), then the set of singular points of X has dimension at most $\dim(X) - 2$. (We say that X is “regular in codimension one.”)

Our next goal is to globalize the construction of integral closure, which in particular will give us a way of desingularizing curves.

The discussion below until the end of §5.3 was not covered on Feb 10 and will be done in the future.

Definition 6.3.3. A morphism of algebraic sets $\phi: Y \rightarrow X$ is **finite** if for every affine open $U \subseteq X$, its preimage $V = \phi^{-1}(U) \subseteq Y$ is affine and the map of rings $\phi^*: \mathcal{O}(U) \rightarrow \mathcal{O}(V)$ is finite (meaning that $\mathcal{O}(V)$ is a finitely generated module over $\mathcal{O}(U)$).

Examples 6.3.4. (a) A closed immersion $i: Y \rightarrow X$ is finite.

- (b) By the “going-up theorem,” a finite map is closed.

- (c) A morphism between affine algebraic sets $\phi: Y \rightarrow X$ is finite if and only if $\phi^*: \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ is finite.
- (d) Let A be a finitely generated domain over k and let A' be its normalization (integral closure in its field of fractions). By “finiteness of integral closure,” the map $\text{MSpec}(A') \rightarrow \text{MSpec}(A)$ is finite.
- (e) By Noether normalization, every affine algebraic set X admits a finite dominant map $X \rightarrow \mathbb{A}^n$ where $n = \dim(X)$.

Corollary 6.3.5. *Let X be a curve. Then there exists a finite and birational map of curves $X' \rightarrow X$ where X' is nonsingular.*

Proof (sketch). If $X = \text{MSpec}(A)$ is affine, we take $X' = \text{MSpec}(A')$ where A' is the normalization of A . In general, cover X by affines $X_i = \text{MSpec}(A_i)$ and set $X'_i = \text{MSpec}(A'_i)$ where A'_i is the normalization of A_i . Then $X_i \cap X_j$ is affine, and its preimage in either X'_i or X'_j is equal to $\text{MSpec}(A'_{ij})$ where A'_{ij} is the normalization of $A_{ij} = \mathcal{O}(X_i \cap X_j)$. This allows us to glue the $X'_i \rightarrow X_i$ into a global $X' \rightarrow X$. \square

Remark 6.3.6. If X is projective then so is X' . We currently lack the tools to show this.

Lemma 6.3.7. *Let C be a non-singular curve. Then every rational map from C to \mathbb{P}^n is everywhere defined.*

Proof. Represent $\phi: C \dashrightarrow \mathbb{P}^n$ by a map $U = C \setminus \{x_1, \dots, x_r\} \rightarrow \mathbb{P}^n$. We may assume U is the largest set on which ϕ is defined, and we aim to show $r = 0$. Otherwise, let $x \in \{x_1, \dots, x_r\}$, and we aim to extend the map over x . This is local around x , and so we may assume that $C = \text{MSpec}(A)$ is affine and $x = V(\pi)$ for some $\pi \in A$ (recall that $\mathcal{O}_{C,x}$ is a DVR), and that there exist $f_0, \dots, f_n \in A[1/\pi]$ which generate the unit ideal such that ϕ is on $D(\pi) = C \setminus \{x\}$ given by $(f_0 : \dots : f_n)$. Clearing denominators, we may assume $f_0, \dots, f_n \in A$ and $f_0 \notin \pi A$. But then $(f_0 : \dots : f_n)$ defines an extension of ϕ to C . \square

Corollary 6.3.8. *Let X and Y be projective non-singular curves. If X and Y are birational, then they are isomorphic.*

6.4. Problem session (Feb 10), part one: blow-ups

Disclaimer: The write-ups from the problem sessions are supposed to be rough transcripts of what has been discussed; they are not meant to be complete.

During the first half of the problem session, we discussed the following construction. The projective space \mathbb{P}^n can be regarded as the space of lines through the origin $0 \in \mathbb{A}^{n+1}$. If $a = (a_0 : \dots : a_n) \in \mathbb{P}^n$ is a point, then the corresponding line $\ell_a \subseteq \mathbb{A}^{n+1}$ is cut out by the system of linear equations

$$a_i T_j - a_j T_i = 0, \quad 0 \leq i < j \leq n$$

(if say $a_0 \neq 0$, then we can consider only the n equations $a_0 T_j - a_j T_0$ with $j = 1, \dots, n$). Note that these are the determinants of the 2×2 minors of the $(n+1) \times 2$ matrix with rows (a_0, \dots, a_n) and (T_0, \dots, T_n) .

The **blowup** of the affine space \mathbb{A}^n is the space of pairs

$$\mathcal{B} = \{(a, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} : x \in L_a\} = V(X_i T_j - X_j T_i) \subseteq \mathbb{P}^n \times \mathbb{A}^{n+1}$$

(where X_0, \dots, X_n are now the homogeneous coordinates on \mathbb{P}^n) which we can regard as the set pairs (L, x) where $x \in \mathbb{A}^{n+1}$ and $L \subseteq \mathbb{A}^{n+1}$ is a line through the origin which contains x . The space \mathcal{B} is thus an algebraic set (in fact, a non-singular variety).

We denote by $\pi: \mathcal{B} \rightarrow \mathbb{A}^{n+1}$ the projection map $(L, x) \mapsto x$. Note that for $x \neq 0$, we have $\pi^{-1}(x) = \{(L_x, x)\}$ where L_x is the unique line through 0 and x . On the other hand, we have $\pi^{-1}(0) = \mathbb{P}^n \times 0$. So

all fibers are singletons except one fiber which has dimension n . The map π induces an isomorphism over $\mathbb{A}^{n+1} \setminus 0$ and hence is birational.

Recall that \mathbb{P}^n is the union of $n + 1$ copies U_0, \dots, U_i of the affine space \mathbb{A}^n . Thus $\mathbb{P}^n \times \mathbb{A}^{n+1}$ is the union of $n + 1$ copies $U_i \times \mathbb{A}^{n+1}$ of \mathbb{A}^{2n+1} , and \mathcal{B} is the union of $n + 1$ open affine subvarieties $\mathcal{B}_i = \mathcal{B} \cap (U_i \times \mathbb{A}^{n+1})$. In order to avoid notational nightmare, let us explicate this in case $n = 1$. Let now the coordinates on \mathbb{A}^2 be x, y and on \mathbb{P}^1 be U, V .

- The coordinates on $\mathcal{B}_0 = \{U \neq 0\}$ are x, y , and $v = V/U$. There is a single equation $xV = yU$ cutting out \mathcal{B} , which translates to $\mathcal{B}_0 = \{xv = y\} \subseteq \mathbb{A}^3$. Note that we have $\mathcal{B}_0 \simeq \mathbb{A}^2$ with coordinates x, v and the map $\pi|_{\mathcal{B}_0}$ is the map $(x, v) \mapsto (x, xv)$.
- Similarly, $\mathcal{B}_1 \simeq \mathbb{A}^2$ with coordinates $y, u = U/V$ with $x = uy$.

See the illustration in Hartshorne I §4.

Definition 6.4.1. Let $X \subseteq \mathbb{A}^{n+1}$ be a closed subset. We define its **strict transform** to be the closed subset $\tilde{X} \subseteq \mathcal{B}$ defined as

$$\tilde{X} = \overline{\pi^{-1}(X \setminus 0)}.$$

The **tangent cone** of X at 0 is $C_0(X) = \tilde{X} \cap \pi^{-1}(0) \subseteq \mathbb{P}^n$.

We computed the strict transform of the following two singular plane curves

$$C : y^2 = x^3, \quad D : y^2 = x^3 + x^2.$$

For this we used the description of the closure from Lecture 1, section 1.5.

Curve C : on \mathcal{B}_0 , we substitute $y = vx$ in the equation of C to get

$$y^2 - x^3 = v^2x^2 - x^3 = x^2(v^2 - x).$$

Since we are computing the closure of this minus the line $\{x = 0\}$, we can divide out the x and conclude that $\tilde{C} \cap \mathcal{B}_0$ is the parabola $x = v^2$. It meets the line $\{x = 0\} = \pi^{-1}(0) \cap \mathcal{B}_0$ in the single point 0 but with multiplicity two (it is tangent to the line). A similar computation in the other chart \mathcal{B}_1 gives that $\tilde{C} \cap \mathcal{B}_1$ is cut out by $1 = u^3y$, which is a smooth curve which does not meet the line $\{y = 0\}$ and in fact \tilde{C} is contained in \mathcal{B}_0 . Conclusion: \tilde{C} is a smooth curve, in fact isomorphic to \mathbb{A}^1 with coordinate $v = y/x$, and $\mathcal{O}(\tilde{C}) = k[v]$ is the normalization of $\mathcal{O}(C) = k[x, y]/(y^2 - x^3) = k[v^2, v^3]$. Its tangent cone consists of one point $(1 : 0)$ with multiplicity two. Geometrically this corresponds to the fact that at 0 , the curve C is singular but seems to have one tangent direction, the horizontal axis $y = 0$.

Curve D : on \mathcal{B}_0 , we substitute $y = vx$ in the equation of D to get

$$y^2 - x^3 - x^2 = x^2(v^2 - x - 1),$$

and again $\tilde{D} \cap \mathcal{B}_0$ is cut out by $x = v^2 - 1$, a parabola meeting $\{x = 0\}$ at two points $v = \pm 1$. Geometrically, these points correspond to the two slopes ± 1 of the two “branches” of D at the origin. We skip the remainder of the computation. Conclusion: \tilde{D} is smooth and its tangent cone consist of two points.

We also noted how for the curve $x^p = y^q$ for a coprime pair (p, q) successive blowups perform the Euclid algorithm on (p, q) , ultimately desingularizing the curve.

6.5. Problem session (Feb 10), part two: What is algebraic geometry?

We discussed the following short list of problems (easy, hard, and unsolved) studied in algebraic geometry. We managed to cover (a)–(d) and part of (e), and we shall return to this review later.

- (a) (Enumerative geometry) How many solutions does the system

$$\begin{cases} f(X, Y) = 0 \\ g(X, Y) = 0 \end{cases}$$

have, for a pair of coprime square-free polynomials $f, g \in k[X, Y]$? The answer (Bezout's theorem) is that this number is at most $\deg(f) \cdot \deg(g)$, and that if k is algebraically closed, it will typically be equal to this bound. More precisely, we will get exactly $\deg(f) \cdot \deg(g)$ if we count the solutions with correct multiplicity, and if we also count points at infinity (in the projective plane \mathbb{P}^2), which correspond to common asymptotes of the curves $f = 0$ and $g = 0$.

For a less trivial question of enumerative kind: how many conics are tangent to five randomly chosen conics in \mathbb{P}^2 ? The answer is 3264 (with probability 1). Why?

- (b) (Real geometry) How many connected components does the set of real solutions of the equation

$$Y^2 = X^3 + aX + b$$

have? Answer: one if $\Delta = 4a^3 + 27b^2 \geq 0$ and two if $\Delta < 0$.

In general, according to Harnack's theorem for an equation $f(X, Y) = 0$ defining a nonsingular curve, the set of real solutions has at most $\binom{\deg(f)}{2} + 1$ connected components.

For a surprisingly still unsolved problem, consider Maxwell's "mystery of point charges" [1]. Consider 3 point charges q_1, q_2, q_3 located at three distinct points $p_i = (x_i, y_i)$ in the plane \mathbb{R}^2 . According to Coulomb's law, the electrostatic force acting charge $c \neq 0$ located at a varying point $p = (x, y)$ equals

$$F(p) = \sum_{i=1}^3 \frac{cq_i(p - p_i)}{|p - p_i|^3}.$$

We say that $p \in \mathbb{R}^2$ is an equilibrium point if $F(p) = 0$. How many equilibrium points can there be?

According to Maxwell, at most four (and in general, for n charges in \mathbb{R}^3 , at most $(n-1)^2$). Nobody knows how Maxwell would justify this conjecture. According to Bezout's theorem and my hands-on calculation, the number of complex solutions is likely equal 1600. Currently, the best known bound, due to Gabrielov–Novikov–Shapiro [2] (2004), is 12.

- (c) (Algebraic topology) What is the "shape" of the set of complex solutions of the equation

$$f(X, Y) = 0?$$

We assume that this set is nonsingular, i.e. that $f, \partial f / \partial X$, and $\partial f / \partial Y$ do not have a common zero, (and, to be completely precise, that the homogenous form of f of highest degree is square-free). The answer is: it is an orientable surface of genus

$$g = \frac{(\deg(f) - 1)(\deg(f) - 2)}{2}$$

with $\deg(f)$ punctures.

- (d) (Number theory) For f as above, how many solutions does $f(X, Y) = 0$ have over \mathbb{Q} ? The answer (Faltings' resolution of the Mordell conjecture): finitely many as long as $g > 1$. In the special case $f(X, Y) = X^n + Y^n - 1$ (with $n \geq 3$) this question is Fermat's last theorem, proved by Wiles and Taylor in 1995.

- (e) (Point counting) How many solutions does the equation

$$E \quad : \quad Y^2 = X^3 - X$$

have over the finite field $\mathbb{F}_7 = \mathbb{Z}/7$? Answer: 7, namely $(0, 0)$, $(\pm 1, 0)$, $(5, \pm 1)$, $(6, \pm 2)$.

What do we learn from this? The equation describes an elliptic curve E , and the solution set over \mathbb{F}_p (with the addition of a single point at infinity, which acts as the neutral element) is a finite abelian group $\bar{E}(\mathbb{F}_p)$, which in this particular case is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/4$. These finite groups are important in cryptography, and their orders $n_p = \#E(\mathbb{F}_p)$ (for a varying prime p) make up in the corresponding L -function

$$L(E, s) = \prod_{p>2} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}, \quad a_p = p + 1 - n_p$$

featured in the **BSD** (Birch and Swinnerton-Dyer) conjecture, which states that the order of vanishing of $L(E, s)$ at $s = 1$. One of the early results in the arithmetic of elliptic curves is the **Hasse bound**

$$|p + 1 - n_p| = |a_p| \leq 2\sqrt{p}.$$

The **Sato–Tate conjecture** (proved in 2011) predicts the distribution of the numbers $a_p/2\sqrt{p}$ in the interval $[-1, 1]$ as $p \rightarrow \infty$.

The Hasse bound was generalized by Schmidt and Weil to arbitrary (smooth and projective) curves over finite fields:

$$|q + 1 - \#C(\mathbb{F}_q)| \leq 2g\sqrt{q}.$$

Curiously, this statement is equivalent to a kind of Riemann hypothesis for the corresponding zeta function $\zeta(C, s)$. We shall prove Weil’s result in the final part of the course.

- (f) (Moduli theory) Describe the set of all (smooth and projective) complex curves of genus $g \geq 2$. Answer (Deligne–Mumford): they are parametrized by an irreducible and smooth algebraic variety of dimension $3g - 3$.
- (g) (Birational geometry) Consider a homogeneous equation of degree $d \geq 1$ in $n + 2 \geq 3$ variables:

$$X \quad : \quad f(X_0, \dots, X_{n+1}) = 0.$$

Let us suppose that the corresponding set of (complex) solutions in \mathbb{P}^{n+1} is non-singular (i.e. the only possible singular point of X in k^{n+2} is the origin). For which values of d and n can the variety X be rational? By definition, this means that there exists a one-to-one map from an open subset of \mathbb{A}^n into X .

It turns out that X is rational for $d \leq 2$, and for $d = 3$ when $n = 2$ (a cubic surface is rational). For $d = 3$ and $n = 3$, the general cubic threefold is not rational, by a celebrated result of Clemens and Griffiths. For $d = 3$ and $n = 4$, the general cubic fourfold is not rational, according to a recent (summer 2025) preprint of Kontsevich, Katzarkov, Pantev, and Yu.

- (h) (Singularity theory) Consider once again a hypersurface (single equation) over \mathbb{C}

$$X \quad : \quad f(T_1, \dots, T_n) = 0$$

and suppose that $f(0, \dots, 0) = 0$ and that $(0, \dots, 0)$ is an isolated point of the zero set of f and all $\partial f / \partial T_i$ (that is, the origin is an isolated singular point of X). What can we say about the topology of the singularity of C around p ? For a beautiful answer, read Milnor’s book *Singular Points of Complex Hypersurfaces*. For a recent development in this area, see J. Fernández de Bobadilla, T. Peřka *Symplectic monodromy at radius zero and equimultiplicity of μ -constant families* (Ann. Math. 2024).

7. Lecture 7 (Feb 12): Sheaves

Recommended reading: Hartshorne II.1, Kempf §4, Vakil §2

7.1. Review of sheaves

We review the basics of sheaf theory, without proofs.

Definition 7.1.1. Let X be a topological space.

(a) A **presheaf** (of abelian groups) \mathcal{F} on X consists of the following data:

- for every open $U \subseteq X$, an abelian group $\mathcal{F}(U)$;
- for every pair of opens $U, V \subseteq X$ such that $V \subseteq U$, a homomorphism $\rho_{U,V}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ (called the *restriction map*)

such that

- i. $\rho_{U,U} = \text{id}$ for every open $U \subseteq X$;
- ii. $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for every triple of opens $U, V, W \subseteq X$ such that $W \subseteq V \subseteq U$.

(In other words, \mathcal{F} is a contravariant functor $\mathcal{F}: \mathbf{Opens}(X)^{\text{op}} \rightarrow \mathbf{Ab}$ from the category (poset) of open subsets of X to the category of abelian groups.)

(b) A **map of presheaves** $f: \mathcal{F} \rightarrow \mathcal{G}$ is a collection of homomorphisms $f_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for every open $U \subseteq X$ which are compatible with the maps in the sense that for $V \subseteq U$ we have

$$f_V \circ \rho_{U,V}^{\mathcal{F}} = \rho_{U,V}^{\mathcal{G}} \circ f_U.$$

(In other words, f is a natural transformation between the two functors $\mathcal{F}, \mathcal{G}: \mathbf{Opens}(X)^{\text{op}} \rightarrow \mathbf{Ab}$.)

(c) We introduce the following notation and terminology.

- We denote the **category of presheaves** on X by $\mathbf{PSh}(X)$.
- Elements of $\mathcal{F}(U)$ are called **sections** of \mathcal{F} over U .
- Elements of $\mathcal{F}(X)$ are called the **global sections** of \mathcal{F} .
- For $s \in \mathcal{F}(U)$ and $V \subseteq U$, we write $s|_V$ for $\rho_{U,V}(s) \in \mathcal{F}(V)$ and call it the **restriction** of s to V .
- For $f: \mathcal{F} \rightarrow \mathcal{G}$ we denote the maps $f_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ simply by f . (In particular, the condition in (b) can be more succinctly written as $f(s|_V) = f(s)|_V$.)

(d) A presheaf \mathcal{F} on X is a **sheaf** if for every open $U \subseteq X$ and every family $\{U_\alpha\}_{\alpha \in I}$ of opens with $U = \bigcup U_\alpha$, and every family of sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that for every $\alpha, \beta \in I$ we have

$$s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta} \quad \text{in } \mathcal{F}(U_\alpha \cap U_\beta)$$

there exists a unique section $s \in \mathcal{F}(U)$ such that for every $\alpha \in I$ we have

$$s|_{U_\alpha} = s_\alpha.$$

We denote by $\mathbf{Sh}(X)$ the category of sheaves (a full subcategory of $\mathbf{PSh}(X)$).

We have defined (pre)sheaves of abelian groups, but in the same way one defines (pre)sheaves of sets, groups, or rings.

Examples 7.1.2. (a) Let X be a topological space and let G be a topological abelian group (e.g. $G = \mathbb{R}$). We define a presheaf \mathcal{C}_G on X by

$$\mathcal{C}_G(U) = \{\text{continuous maps } U \rightarrow G\}$$

(with the obvious restriction maps). Then \mathcal{C}_G is a sheaf.

- (b) Special case of (1): let G be an abelian group, and make it into a topological group by giving it the discrete topology. Then $\mathcal{C}_G(U)$ consists of *locally constant* functions $U \rightarrow G$, and the sheaf \mathcal{C}_G is called the **constant sheaf** with value G and denoted by \underline{G} .
- (c) Another special case of (1): let G be an abelian group, and this time make it into a topological group by giving it the indiscrete topology. Then $\mathcal{C}_G(U)$ consists of all maps of sets $U \rightarrow G$.
- (d) Let $X = \mathbb{C}$ be the complex plane and let $\mathcal{O}(U)$ denote the ring of holomorphic functions on an open $U \subseteq X$. This is a sheaf of rings on X .
- (e) Let X be a space with functions (over an implicitly chosen field k). Then the association $U \mapsto \mathcal{O}(U) = \{\text{regular functions on } U\}$ defines a sheaf on X , denoted by \mathcal{O}_X and called its **structure sheaf**.
- (f) Let $f: Y \rightarrow X$ be a map of topological spaces. Let \mathcal{S}_f be the presheaf associating to $U \subseteq X$ the set of all sections of $f|_U$, i.e. of all continuous maps $s: U \rightarrow Y$ such that the composition $U \rightarrow Y \rightarrow X$ equals the inclusion map $U \rightarrow X$. This is a sheaf of sets on X , called the **sheaf of sections** of the map f .

Definition 7.1.3. Let X be a topological space.

- (a) Let \mathcal{F} be a presheaf on X and let $x \in X$. We define the **stalk** of \mathcal{F} at x to be the direct limit

$$\mathcal{F}_x = \varinjlim_U \mathcal{F}(U)$$

over all open neighborhoods U of x . (Concretely, elements of \mathcal{F}_x are represented by pairs (U, s) where $x \in U$ and $s \in \mathcal{F}(U)$, where we identify (U, s) with (U', s') if there exists an open $U'' \subseteq U \cap U'$ with $x \in U''$ such that $s|_{U''} = s'|_{U''}$. We denote the image of (U, s) in \mathcal{F}_x by s_x and call it the **germ** of s at x .)

- (b) The **sheafification** of \mathcal{F} is the presheaf \mathcal{F}^a given by associating to $U \subseteq X$ the set of all functions $s: U \rightarrow \bigsqcup_{x \in U} \mathcal{F}_x$ such that
- i. $s(x) \in \mathcal{F}_x$ for every $x \in U$;
 - ii. for every $x \in U$ there exists an open neighborhood $V \subseteq U$ of x and a section $t \in \mathcal{F}(V)$ such that for every $y \in V$, we have $s(y) = t_y$.

Note that \mathcal{F}^a is a sub-presheaf of $D(\mathcal{F})$, and that there is a map of presheaves $i: \mathcal{F} \rightarrow \mathcal{F}^a$ mapping $s \in \mathcal{F}(U)$ to the function $x \mapsto s_x$. These constructions are functorial in \mathcal{F} in the obvious way.

Proposition 7.1.4. Let \mathcal{F} be a presheaf on a topological space X .

- (a) The presheaf $D(\mathcal{F})$ is a sheaf.
- (b) The presheaf \mathcal{F}^a is a subsheaf of $D(\mathcal{F})$.
- (c) Every map $f: \mathcal{F} \rightarrow \mathcal{G}$ into a sheaf \mathcal{G} factors uniquely through $i: \mathcal{F} \rightarrow \mathcal{F}^a$.

(In other words, $\mathcal{F} \mapsto \mathcal{F}^a$ defines a functor $\mathbf{PSh}(X) \rightarrow \mathbf{Sh}(X)$, called the **sheafification functor**, which is left adjoint to the inclusion functor $\mathbf{Sh}(X) \rightarrow \mathbf{PSh}(X)$.)

(d) \mathcal{F} is a sheaf if and only if the map $i: \mathcal{F} \rightarrow \mathcal{F}^a$ is an isomorphism.

Lemma 7.1.5. A map of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism if and only if for every $x \in X$, the map on stalks $\mathcal{F}_x \rightarrow \mathcal{G}_x$ is an isomorphism.

Operations on sheaves. Let X be a topological space.

- (a) **Binary coproducts.** Let \mathcal{F} and \mathcal{G} be sheaves on X . Then $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf (denoted $\mathcal{F} \oplus \mathcal{G}$) which is the categorical coproduct of \mathcal{F} and \mathcal{G} . It coincides with the binary product $\mathcal{F} \times \mathcal{G}$, and for $x \in X$ we have $(\mathcal{F} \oplus \mathcal{G})_x = \mathcal{F}_x \oplus \mathcal{G}_x$.
- (b) **Kernels.** Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a map of sheaves on X . Then $U \mapsto \ker(\mathcal{F}(U) \rightarrow \mathcal{G}(U))$ is a subsheaf of \mathcal{F} denoted by $\ker(f)$. It is the kernel (equalizer of f and the zero map) in the category of sheaves on X , and for $x \in X$ we have $\ker(f)_x = \ker(\mathcal{F}_x \rightarrow \mathcal{G}_x)$.
- (c) **Products.** Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a family of sheaves on X . Then $U \mapsto \prod_{\alpha \in I} \mathcal{F}_\alpha(U)$ is a sheaf on X , which is the categorical product $\prod_{\alpha \in I} \mathcal{F}_\alpha$ of the family of sheaves $\{\mathcal{F}_\alpha\}_{\alpha \in I}$. In general, for $x \in X$, $(\prod_{\alpha \in I} \mathcal{F}_\alpha)_x$ does not coincide with $\prod_{\alpha \in I} (\mathcal{F}_\alpha)_x$.
- (d) **Cokernels.** Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a map of sheaves on X . The presheaf $\text{cok}_{\text{pre}}(f)$ defined by $U \mapsto \text{cok}(\mathcal{F}(U) \rightarrow \mathcal{G}(U))$ is not a sheaf in general. We define the cokernel $\text{cok}(f) = \text{cok}_{\text{pre}}(f)^a$ to be the sheafification of this presheaf. It is the categorical cokernel of f (the coequalizer of f and the zero map) and for every $x \in X$, we have $\text{cok}(f)_x = \text{cok}(\mathcal{F}_x \rightarrow \mathcal{G}_x)$.
- (e) **Quotient.** For a sheaf \mathcal{G} and a subsheaf $\mathcal{F} \subseteq \mathcal{G}$, the quotient \mathcal{G}/\mathcal{F} is the cokernel of the inclusion $\mathcal{F} \rightarrow \mathcal{G}$, i.e. the sheafification of the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$.
- (f) **Image.** The image $\text{im}(f)$ of a map of sheaves $f: \mathcal{F} \rightarrow \mathcal{G}$ is the sheafification of the presheaf $U \mapsto \text{im}(\mathcal{F}(U) \rightarrow \mathcal{G}(U))$ (which is a subsheaf of \mathcal{G}), or equivalently the quotient $\mathcal{F}/\ker(f)$ or the kernel of $\mathcal{G} \rightarrow \text{cok}(\mathcal{F})$. (In particular, the category $\mathbf{Sh}(X)$ is an abelian category.)
- (g) **Cohomology and exactness.** A complex of sheaves \mathcal{F}^\bullet is a sequence of sheaves \mathcal{F}^n ($n \in \mathbb{Z}$) and maps $d^n: \mathcal{F}^n \rightarrow \mathcal{F}^{n+1}$ such that $d^n \circ d^{n-1} = 0$ for all n . Its n -th cohomology sheaf $\mathcal{H}^n(\mathcal{F}^\bullet)$ is the quotient $\ker(d^n)/\text{im}(d^{n-1})$. We say that \mathcal{F}^\bullet is exact if $\mathcal{H}^n(\mathcal{F}^\bullet) = 0$ for all n . The stalk $\mathcal{H}^n(\mathcal{F}^\bullet)_x$ is the n -th cohomology of the complex of abelian groups \mathcal{F}_x^\bullet , and \mathcal{F}^\bullet is exact if and only if the complexes \mathcal{F}_x^\bullet are exact for all $x \in X$.
- (h) **Coproducts.** Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a family of sheaves on X . Then the presheaf $U \mapsto \bigoplus_{\alpha \in I} \mathcal{F}_\alpha(U)$ is not a sheaf in general, and we define $\bigoplus_{\alpha \in I} \mathcal{F}_\alpha$ to be its sheafification, which is the categorical coproduct of the family of sheaves $\{\mathcal{F}_\alpha\}_{\alpha \in I}$.
- (i) **Restriction.** Let \mathcal{F} be a sheaf on X and let $U \subseteq X$ be an open subset. Then the presheaf on U given by $V \mapsto \mathcal{F}(V)$ (for opens $V \subseteq U$) is a sheaf on U , denoted by $\mathcal{F}|_U$.
- (j) **Tensor product.** Let A be a ring and let \mathcal{F} and \mathcal{G} be sheaves of A -modules on X . The presheaf $U \mapsto \mathcal{F}(U) \otimes_A \mathcal{G}(U)$ is in general not a sheaf, and we define $\mathcal{F} \otimes_A \mathcal{G}$ to be its sheafification. There is an A -bilinear map $\mathcal{F} \times \mathcal{G} \rightarrow \mathcal{F} \otimes_A \mathcal{G}$ sending $(s, t) \in \mathcal{F}(U) \times \mathcal{G}(U)$ to the image of $s \otimes t \in \mathcal{F}(U) \otimes_A \mathcal{G}(U)$ in $(\mathcal{F} \otimes_A \mathcal{G})(U)$, which has the usual universal property: every A -bilinear map $\mathcal{F} \times \mathcal{G} \rightarrow \mathcal{H}$ into a sheaf of A -modules \mathcal{H} factors uniquely through an A -linear map $\mathcal{F} \otimes_A \mathcal{G} \rightarrow \mathcal{H}$. Moreover, we have $(\mathcal{F} \otimes_A \mathcal{G})_x = \mathcal{F}_x \otimes_A \mathcal{G}_x$.
- (k) **Sheaf of homomorphisms.** Let \mathcal{F} and \mathcal{G} be sheaves of A -modules on X (there are obvious variants for sheaves of sets etc.). Then the presheaf

$$U \mapsto \text{Hom}_A(\mathcal{F}|_U, \mathcal{G}|_U)$$

sending U to the A -module of all A -linear maps $\mathcal{F}|_U \rightarrow \mathcal{G}|_U$ is a sheaf denoted by $\mathcal{H}om_A(\mathcal{F}, \mathcal{G})$. It is right-adjoint to \otimes_A in the sense that

$$\mathrm{Hom}_A(\mathcal{F} \otimes_A \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}_A(\mathcal{F}, \mathcal{H}om_A(\mathcal{G}, \mathcal{H})).$$

- (l) **Direct image.** Let $\phi : Y \rightarrow X$ be a map of topological spaces and let \mathcal{G} be a sheaf on Y . Then the presheaf on X defined by $U \mapsto \mathcal{G}(\phi^{-1}(U))$ is a sheaf denoted by $\phi_*(\mathcal{G})$.
- (m) **Inverse image.** Let $\phi : Y \rightarrow X$ be a map of topological spaces and let \mathcal{F} be a sheaf on X . We define $\phi^*(\mathcal{F})$ to be the sheafification of the following presheaf on Y :

$$U \mapsto \varinjlim_{\phi(U) \subseteq V} \mathcal{F}(V)$$

(filtered colimit over all opens $V \subseteq X$ containing the image $\phi(U)$). For example, the stalk \mathcal{F}_x is the pull-back $i_x^*(\mathcal{F})$ along the inclusion $i_x : \{x\} \rightarrow X$. In general, for every $y \in Y$, we have a canonical identification

$$(\phi^*\mathcal{F})_y = \mathcal{F}_{\phi(y)}.$$

The functors ϕ_* and ϕ^* form an adjoint pair:

$$\mathrm{Hom}(\phi^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, \phi_*\mathcal{G}).$$

Moreover, ϕ_* is a left exact functor and ϕ^* is exact.

8. Lecture 8 (Feb 17): Schemes

Recommended reading: Hartshorne II.2–3

8.1. Sheaves on a base

Let X be a topological space. A **base** of the topology on X is a family of open subsets $\mathcal{B} \subseteq \mathbf{Opens}(X)$ such that every open of X is the union of a family of elements of \mathcal{B} .

Examples 8.1.1. (a) Let $X = \mathbb{R}^n$ and let \mathcal{B} be the family of all open balls.

(b) Let X be an algebraic set and let \mathcal{B} be the family of all affine opens.

(c) Let $X = \text{Spec}(A)$ for a ring A and let \mathcal{B} be the family of all open subsets of the form $D(f)$ for $f \in A$.

Definition 8.1.2 (Sheaf on \mathcal{B}). Let X be a topological space and let \mathcal{B} be a base of its topology. A **sheaf** on \mathcal{B} is a functor $\mathcal{F}: \mathcal{B}^{\text{op}} \rightarrow \mathbf{Ab}$ such that for every $U \in \mathcal{B}$ and every family $\{U_\alpha\}_{\alpha \in I}$ of elements of \mathcal{B} such that $U = \bigcup_{\alpha \in I} U_\alpha$, and every family of sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that for every $\alpha, \beta \in I$ and every $V \in \mathcal{B}$ contained in $U_\alpha \cap U_\beta$ we have

$$s_\alpha|_V = s_\beta|_V$$

there exists a unique section $s \in \mathcal{F}(U)$ such that for every $\alpha \in I$ we have

$$s|_{U_\alpha} = s_\alpha.$$

Remark 8.1.3. Suppose that \mathcal{B} is closed under intersections (as is the case in Example 8.1.1(b) (if X is separated) and Example 8.1.1(c)). In this case, we can take $V = U_\alpha \cap U_\beta$ in the above definition.

Proposition 8.1.4. Let X be a topological space and let \mathcal{B} be a base of its topology. Then the restriction functor $\mathbf{PSh}(X) \rightarrow \mathbf{PSh}(\mathcal{B})$ induces an equivalence of categories

$$\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathcal{B}).$$

Proof (sketch). We first note that by definition of a base, every open neighborhood of a point $x \in X$ contains a neighborhood which belongs to the base \mathcal{B} . Therefore, the stalk \mathcal{F}_x of a presheaf \mathcal{F} on X depends only on its restriction to \mathcal{B} :

$$\mathcal{F}_x = \varinjlim_{x \in U \in \mathcal{B}} \mathcal{F}(U).$$

The same formula defines the stalk \mathcal{F}_x of a presheaf \mathcal{F} on \mathcal{B} . Our construction of the sheafification (Lecture 7) can therefore be adapted to give a functor from presheaves on \mathcal{B} to sheaves on X . That is, for a presheaf \mathcal{F} on \mathcal{B} we define the presheaf \mathcal{F}^a on X associating to $U \subseteq X$ the set of all $(s(x)) \in \prod_{x \in U} \mathcal{F}_x$ such that for every $x \in U$ there exists an open neighborhood $V \subseteq X$ of x such that $V \in \mathcal{B}$ and a section $t \in \mathcal{F}(V)$ such that for every $y \in V$ we have $s(y) = t_y$. The proof of the existence and basic properties of sheafification adapts easily to finish the current proof. \square

8.2. Basic lemma

Lemma 8.2.1. Let A be a ring and let $f_1, \dots, f_r \in A$ be elements generating the unit ideal in A . Then the map $A \mapsto \prod_{i=1}^r A[f_i^{-1}]$ sending $g \in A$ to (g, \dots, g) induces an isomorphism

$$A \xrightarrow{\simeq} \ker \left(\prod_{i=1}^r A[f_i^{-1}] \xrightarrow{\delta} \prod_{i,j=1}^r A[(f_i f_j)^{-1}] \right)$$

where the map δ is defined by

$$\delta(g_1, \dots, g_r)_{ij} = g_i - g_j \in A[(f_i f_j)^{-1}].$$

Proof. We handle injectivity first. Suppose that $g \in A$ maps to zero in $\prod_{i=1}^r A[f_i^{-1}]$. This means that for each $i = 1, \dots, r$ we have $f_i^{n_i} g = 0$ for some $n_i \geq 1$. Now $A = (f_1, \dots, f_r)$ implies $A = (f_1^{n_1}, \dots, f_r^{n_r})$. Write $1 = \sum_{i=1}^r h_i f_i^{n_i}$ for some $h_1, \dots, h_r \in A$. Multiply the last equality by g to get

$$g = \sum_{i=1}^r h_i f_i^{n_i} g = 0.$$

Surjectivity is easily seen to be equivalent to the following claim: given $g_1, \dots, g_r \in A$ and an integers $n, m \geq 0$ such that

$$(f_i f_j)^m (f_i^n g_j - f_j^n g_i) = 0$$

for all $i, j \in \{1, \dots, r\}$, there exists an integer $k \geq 0$ such that

$$f_i^k (g f_i^n - g_i) = 0$$

for all $i = 1, \dots, r$. We will show this with $k = m$. Now $A = (f_1, \dots, f_r)$ implies $A = (f_1^{n+m}, \dots, f_r^{n+m})$, so we can write $1 = \sum_{j=1}^r h_j f_j^{n+m}$ for some $h_1, \dots, h_r \in A$. Let

$$g = \sum_{j=1}^r h_j f_j^n g_j.$$

Then for $i = 1, \dots, r$ we have

$$g_i f_i^m = \sum_{j=1}^r h_j f_j^{n+m} g_i f_i^m = \sum_{j=1}^r h_j f_i^{n+m} g_j f_j^m = f_i^{n+m} \sum_{j=1}^r h_j f_j^n g_j = f_i^{n+m} g,$$

as desired. \square

Corollary 8.2.2. *Let A be a ring and let $X = \text{Spec}(A)$. There exists a unique sheaf of rings \mathcal{O}_X on X such that for every $f \in A$ we have*

$$\mathcal{O}_X(D(f)) = A[f^{-1}]$$

(and such that for $f, g \in A$, the restriction map $\mathcal{O}_X(D(f)) \rightarrow \mathcal{O}_X(D(fg))$ coincides with the natural map $A[f^{-1}] \rightarrow A[(fg)^{-1}]$). Moreover, for every $x \in X$ corresponding to a prime ideal $\mathfrak{p}_x \subseteq A$, we have

$$\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}_x}.$$

Proof. We first check that the ring $A[f^{-1}]$ depends only on the open subset $U = D(f)$. To this end, let

$$S_U = \{g \in A : D(g) \subseteq U\} = \bigcap_{x \in U} (A \setminus \mathfrak{p}_x).$$

Then $A[f^{-1}] = A[S_U^{-1}]$.

Now consider the base \mathcal{B} of the topology on X consisting of all opens of the form $D(f)$. The association $D(f) \mapsto A[f^{-1}]$ makes sense thanks to the previous paragraph, and defines a presheaf on the base \mathcal{B} .

We now use Lemma 8.2.1 to check that this presheaf is a sheaf on the base \mathcal{B} . Let thus $U = D(f)$ be covered by $U_\alpha = D(f_\alpha)$, and let $s_\alpha \in A[f_\alpha^{-1}]$ be elements such that for every $\alpha, \beta \in I$, the images of s_α and s_β in $A[(f_\alpha f_\beta)^{-1}]$ are equal. Since $D(f) \simeq \text{Spec}(A[f^{-1}])$ is quasi-compact, there exists a finite subset $I_0 = \{1, \dots, r\} \subseteq I$ such that $U = \bigcup_{i=1}^r U_i$. In other words, f_1, \dots, f_r generate the unit ideal in $A[f^{-1}]$. We are now in position to apply Lemma 8.2.1 to the ring $A[f^{-1}]$, obtaining a unique $s \in A[f^{-1}]$ such that s_i is the image of s in $A[f_i^{-1}]$ for $i = 1, \dots, r$.

Since possibly $I_0 \neq I$, we still need to verify that s_α is the image of s in $A[f_\alpha^{-1}]$ for all $\alpha \in I$. But this is easy, simply label α as $r+1$ and run the previous argument for $I_1 = I \cup \{\alpha\} = \{1, \dots, r, r+1\}$. \square

8.3. Schemes

Definition 8.3.1. (1) A **ringed space** is a pair (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is a sheaf of rings on X . (Often, we shall write X meaning the pair (X, \mathcal{O}_X) .)

- (2) A **morphism** of ringed spaces $(Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ is a pair (ϕ, ϕ^*) where $\phi: Y \rightarrow X$ is a continuous map and where $\phi^*: \mathcal{O}_X \rightarrow \phi_*\mathcal{O}_Y$ is a homomorphism of sheaves of rings on X (in other words, a compatible system of ring homomorphisms

$$\phi^*: \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(\phi^{-1}(U)).$$

(Again, we shall simply write $\phi: Y \rightarrow X$ instead of $(\phi, \phi^*): (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$.)

- (3) A **locally ringed space** is a ringed space (X, \mathcal{O}_X) such that for every $x \in X$, the stalk $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$ is a local ring.
- (4) A **morphism** of locally ringed spaces $(Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ is a map of ringed spaces $\phi: Y \rightarrow X$ such that for every $y \in Y$, the induced map

$$\phi^*: \mathcal{O}_{X,\phi(x)} \longrightarrow \mathcal{O}_{Y,y}$$

is a local ring homomorphism (maps the maximal ideal into the maximal ideal).

Examples 8.3.2. 1. Let X be a topological space and let A be a ring. Then the pair (X, \underline{A}) of X with the constant sheaf with value A is a ringed space which is a locally ringed space if and only if A is local or $X = \emptyset$.

2. Every swf can be regarded as a locally ringed space, by forgetting that the structure sheaf consists of k -valued functions. Morphisms of swf's induce morphisms of locally ringed spaces. (See Problem Set 4.)
3. Let $X = \mathbb{C}$ be the complex plane and let \mathcal{O}_X be the sheaf of holomorphic functions on X . Then X is a locally ringed space (this is a special case of (2)).
4. Let $X = \text{Spec}(A)$ for a ring A , and let \mathcal{O}_X be the structure sheaf constructed in Corollary 8.2.2. Then (X, \mathcal{O}_X) is a locally ringed space.

Let X be a locally ringed space. We introduce the following notation, familiar from our treatment of $\text{Spec}(A)$ last semester:

- for $x \in X$, we denote by $\kappa(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$ and call it the **residue field** of x ;
- for an open $U \subseteq X$, an $f \in \mathcal{O}_X(U)$, and $x \in U$, we denote by $f(x)$ the image of $f_x \in \mathcal{O}_{X,x}$ in $\kappa(x)$.

We note that the notion of a map of locally ringed spaces $\phi: Y \rightarrow X$ is rigged so that for $y \in Y$ and an open neighborhood U of $x = \phi(y)$ we have a unique map of residue fields $\kappa(x) \rightarrow \kappa(y)$ fitting inside the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_Y(\phi^{-1}(U)) & \longrightarrow & \mathcal{O}_{Y,y} & \longrightarrow & \kappa(y) \\ \phi^* \uparrow & & \phi^* \uparrow & & \uparrow \text{---} \\ \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_{X,x} & \longrightarrow & \kappa(x). \end{array}$$

Thus, implicitly treating $\kappa(x)$ as a subfield of $\kappa(y)$ via this map, we have the equality in $\kappa(y)$:

$$\phi^*(f)(y) = f(\phi(y)),$$

which lets us pretend that ϕ^* is defined by composing regular functions with ϕ , as in the case of swf's.

Lemma 8.3.3. *Let X be a locally ringed space, let $U \subseteq X$ be an open subset, and let $f \in \mathcal{O}_X(U)$. Then the subset*

$$D(f) = \{x \in U : f(x) \neq 0\} \subseteq U$$

is open and $f|_{D(f)} \in \mathcal{O}_X(D(f))^\times$.

Proof. If $x \in D(f)$, then $f_x = (U, f) \in \mathcal{O}_{X,x}$ is not in the maximal ideal, and hence is invertible. Therefore there exists a germ (V, g) (where $x \in V$ and $g \in \mathcal{O}_X(V)$) such that $(U, f) \cdot (V, g) = 1$. This equality means that there exists a $W \subseteq U \cap V$ containing x such that $f|_W \cdot g|_W = 1$ in $\mathcal{O}_X(W)$. But then for every $y \in W$, we have $f_y \cdot g_y = 1$ in $\mathcal{O}_{X,y}$, so that $f_y \in \mathcal{O}_{X,y}$ is invertible, i.e. $f(y) \neq 0$. This shows that $x \in W \subseteq D(f)$, and hence that $D(f)$ is open (as x was arbitrary). Moreover, we have constructed an open cover $D(f) = \bigcup_{\alpha \in I} W_\alpha$ and elements $g_\alpha \in \mathcal{O}_X(W_\alpha)$ such that $f|_{W_\alpha} \cdot g_\alpha = 1$. These elements form a compatible family: on $W_\alpha \cap W_\beta$, both g_α and g_β give inverses of $f|_{W_\alpha \cap W_\beta}$ and hence are equal. By the sheaf condition, we have a unique $g \in \mathcal{O}_X(D(f))$ with $g_\alpha = g|_{W_\alpha}$. Then $f|_{D(f)} \cdot g = 1$, as checked after restriction to each W_α . \square

Definition 8.3.4. (a) An **affine scheme** is a locally ringed space (X, \mathcal{O}_X) isomorphic to $(\text{Spec}(A), \mathcal{O}_{\text{Spec}A})$ for some ring A (note that then $A = \mathcal{O}_X(X)$).

(b) A **scheme** is a locally ringed space (X, \mathcal{O}_X) which is locally isomorphic to an affine scheme; more precisely, there exists an open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that each $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine scheme.

(c) A morphism of schemes is a morphism of locally ringed spaces.

Lemma 8.3.5. *Let A be a ring, let $X = \text{Spec}(A)$, and let $f \in A$. Then*

$$(D(f), \mathcal{O}_X|_{D(f)}) \simeq \text{Spec}(A[f^{-1}])$$

is an affine scheme as well.

Proof. The identification of underlying spaces is known from commutative algebra, and the identification of structure sheaves is obvious. \square

Corollary 8.3.6. *Let X be a scheme.*

(a) *Every open subspace of X (endowed with the restriction of \mathcal{O}_X) is a scheme.*

(b) *Affine open subsets form a base of the topology on X .*

Proof. (a) Let $V \subseteq X$ be an open subspace and let $X = \bigcup U_\alpha$ be an affine open cover. Then $V = \bigcup (U_\alpha \cap V)$. However, the $U_\alpha \cap V$ might not be affine. To deal with this, note that since $U_\alpha \cap V$ is an open of $U_\alpha = \text{Spec}(A_\alpha)$, $A_\alpha = \mathcal{O}_X(U_\alpha)$, it is covered by standard opens $W_{\alpha\beta} = D(f_{\alpha\beta})$ which are isomorphic to $\text{Spec}(A_\alpha[f_{\alpha\beta}^{-1}])$ by the previous lemma.

(b) Again let $X = \bigcup U_\alpha$ be an affine open cover. Then for every α the standard opens $D(f) \simeq \text{Spec}(\mathcal{O}(U_\alpha)[f^{-1}]) \subseteq U_\alpha$, $f \in \mathcal{O}(U_\alpha)$ form a basis of the topology on U_α , and hence varying α and f we obtain a base of affine opens on X . \square

We will prove the following universal property of $\text{Spec}(A)$ next time:

Proposition 8.3.7. *Let X be an affine scheme and let Y be a locally ringed space. Then the map*

$$\phi \mapsto \phi^* : \text{Hom}_{\text{LRS}}(Y, X) \longrightarrow \text{Hom}_{\text{Rings}}(\mathcal{O}(X), \mathcal{O}(Y))$$

is bijective.

Corollary 8.3.8. *The functor $A \mapsto \text{Spec}(A)$ establishes an arrow-reversing equivalence between commutative rings and affine schemes, with inverse functor $X \mapsto \mathcal{O}_X(X)$.*

8.4. Problem session (Feb 17)

We discussed the following problems.

1. Prove Lemma 8.3.3.

Done, see the proof of Lemma 8.3.3.

2. More motivation for the definition of a morphism of LRS [Hartshorne, Example II 2.3.2]

Let R be a discrete valuation ring, and write $X = \text{Spec}(R) = \{\eta, s\}$ where s is closed. Let $K = \text{Frac}(R)$ and $Y = \text{Spec}(K) = \{y\}$. The map $R \rightarrow K$ induces a map of schemes $\phi: Y \rightarrow X$ sending y to η . We constructed a different map of ringed spaces (but not on locally ringed spaces) $\psi: Y \rightarrow X$ which sends y to s . This shows the importance of the extra condition in the definition of a morphism of locally ringed spaces, and that the assertion of Proposition 8.3.7 is false if we take maps of ringed spaces on the source.

3. Tangent vectors as maps from a fat point [Hartshorne Exercise II 2.8]

As one motivation for the notion of a scheme we show a very geometric way of treating tangent vectors, made possible by the admission of nilpotents.

Let k be an algebraically closed field and let A be a finitely generated and reduced k -algebra. Let $x \in \text{MSpec}(A) = \text{Hom}_k(A, k)$ be a point of the corresponding affine algebraic set. The **Zariski cotangent space** at x is the vector space over k

$$T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee = \text{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k).$$

Let $X = \text{Spec}(A)$, $T = \text{Spec}(k[\varepsilon]/(\varepsilon^2))$ and $\star = \text{Spec}(k)$. Let $\star \rightarrow X$ be the map induced by $\text{ev}_x: A \rightarrow k$ and let $\star \rightarrow T$ be the map induced by $\varepsilon \mapsto 0$. Consider the problem of finding a map $v: T \rightarrow X$ of k -schemes fitting inside the commutative diagram

$$\begin{array}{ccc} \star & \longrightarrow & X \\ \downarrow & & \nearrow \\ T & & \end{array}$$

In other words, we seek maps from the “fat point” $T \rightarrow X$ which send the underlying “classical” point $\star \subseteq T$ to x .

Claim. Such maps v are in bijection with the Zariski cotangent space T_x .

To see this, we use the equivalence between affine schemes and rings to translate the diagram into a diagram of k -algebras

$$\begin{array}{ccc} k & \xleftarrow{\text{ev}_x} & A \\ \varepsilon \mapsto 0 \uparrow & & \nearrow \\ k[\varepsilon]/(\varepsilon^2) & & \end{array}$$

The diagonal map $v^*: A \rightarrow k[\varepsilon]/(\varepsilon^2)$ sends thus an element $f \in A$ to

$$v^*(f) = f(x) + \delta_v(f)\varepsilon$$

for some map $\delta_v(f): A \rightarrow k$. We verify easily that v^* is a ring homomorphism if and only if $\delta_v(f)$ is derivation, where the target k is regarded as the A -module A/\mathfrak{m}_x .

It remains to identify $\text{Der}_k(A, k)$ with $T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. To this end, we have a direct sum decomposition $A \simeq k \oplus \mathfrak{m}_x$ of A as a k -vector space (decomposing $f \in A$ as $f(x) + (f - f(x))$). A k -linear

derivation $\delta: A \rightarrow k$ vanishes on k , and hence is determined by its restriction $\mathfrak{m}_x \rightarrow k$. Further, the Leibniz rule

$$\delta(fg) = f(x) \cdot \delta(g) + g(x) \cdot \delta(f)$$

shows that δ vanishes on \mathfrak{m}_x^2 . We thus obtain a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow k$ (i.e. an element of T_x) which uniquely determines δ .

4. *Reduced schemes.* Let X be a scheme. Show that the following conditions are equivalent:

- (a) X is reduced, i.e. for every open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced,
- (b) for every affine open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced,
- (c) there exists an affine open cover $X = \bigcup U_\alpha$ where $\mathcal{O}_X(U_\alpha)$ is reduced for each α ,
- (d) for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is reduced.

The implications (a) \Rightarrow (b) \Rightarrow (c) are obvious. To show (c) \Rightarrow (d), let $x \in X$ and let α be such that $x \in U_\alpha$. If $\mathfrak{p}_x \subseteq \mathcal{O}(U_\alpha)$ is the corresponding prime ideal, we have

$$\mathcal{O}_{X,x} = \mathcal{O}_{U_\alpha,x} = \mathcal{O}(U_\alpha)_{\mathfrak{p}_x}.$$

Now, a localization $A[S^{-1}]$ of a reduced ring is reduced: if $(a/s)^n = 0$, we have $ta^n = 0$ for some $t \in S$, and then $(ta)^n = t^{n-1} \cdot ta^n = 0$, implying $ta = 0$, so $a/s = 0$. Thus $\mathcal{O}_{X,x}$ is reduced. Finally, to show (d) \Rightarrow (a), note that we have an injection

$$\mathcal{O}_X(U) \longrightarrow \prod_{x \in U} \mathcal{O}_{X,x}$$

and a subring of the product of a family of reduced rings is reduced.

9. Lecture 9 (Feb 19): More on schemes

Recommended reading: Hartshorne II.2–3

9.1. Exactness of the Čech and Amitsur complexes

In Lecture 8, in the course of constructing the structure sheaf on an affine scheme $\text{Spec}(A)$, we gave a slightly tedious proof that for $f_1, \dots, f_r \in A$ generating the unit ideal, the sequence

$$0 \longrightarrow A \longrightarrow \prod_{i=1}^r A[f_i^{-1}] \longrightarrow \prod_{i,j=1}^r A[(f_i f_j)^{-1}]$$

is exact (see Lemma 8.2.1). The following result generalizes this in two ways: we replace A and its localizations by an A -module M and its localizations, and we extend the sequence to the right. Curiously, the proof does not involve tedious calculations but rather general ideas surrounding the notion of a faithfully flat morphism (introduced below).

Lemma 9.1.1. *Let A be a ring, let $f_1, \dots, f_r \in A$ with $A = (f_1, \dots, f_r)$, and let M be an A -module. Then, the following complex (called the augmented Čech complex of M) is exact*

$$0 \longrightarrow M \longrightarrow \prod_{i_0} M[f_{i_0}^{-1}] \longrightarrow \prod_{i_0, i_1} M[(f_{i_0} f_{i_1})^{-1}] \longrightarrow \dots$$

(Here M is put in degree -1 and the differential

$$d: \prod_{i_0, \dots, i_p} M[(f_{i_0} \dots f_{i_p})^{-1}] \longrightarrow \prod_{i_0, \dots, i_p, i_{p+1}} M[(f_{i_0} \dots f_{i_p} f_{i_{p+1}})^{-1}]$$

is given by the formula

$$d((m_{i_0 \dots i_p}))_{j_0 \dots j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k m_{j_0 \dots \widehat{j_k} \dots j_{p+1}}$$

where $\widehat{j_k}$ means that j_k is omitted.)

Observations. We precede the proof with a sequence of observations.

- (1) For a ring A and $f, g \in A$, we have

$$A[f^{-1}] \otimes_A A[g^{-1}] \simeq A[(fg)^{-1}]$$

(universal properties give the most immediate proof of this.)

- (2) Let $B = \prod_{i=1}^r A[f_i^{-1}]$. Then the p -th term in the augmented Čech complex of M can be written as

$$\prod_{i_0, \dots, i_p} M[(f_{i_0} \dots f_{i_p})^{-1}] \simeq \underbrace{B \otimes B \otimes \dots \otimes B}_{p+1 \text{ times}} \otimes M$$

(all \otimes are \otimes_A). The differentials in this complex are given by the formula

$$d(b_0 \otimes \dots \otimes b_p \otimes m) = \sum_{k=0}^p (-1)^k b_0 \otimes \dots \otimes 1 \otimes \dots \otimes b_p \otimes m \quad (\text{introduce } 1 \otimes \text{ before } b_k).$$

This motivates us to make the following definition: let $A \rightarrow B$ be a map of rings and let M be an A -module. Then the augmented **Amitsur complex** of M is the complex

$$C_a^\bullet(B/A, M) = \left[0 \longrightarrow M \longrightarrow B \otimes M \longrightarrow B \otimes B \otimes M \longrightarrow \dots \right]$$

with M in degree -1 and differential given by the formula above. Thus the augmented Čech complex is the augmented Amitsur complex for $B = \prod_{i=1}^r A[f_i^{-1}]$.

- (3) The formation of the augmented Amitsur complex is compatible with extension of scalars in the following way. Let $A \rightarrow B$ be a map of rings, let M be an A -module, and let $A \rightarrow A'$ be a map of rings. Let $B' = A' \otimes_A B$ and $M' = A' \otimes_A M$. Then

$$C_a^\bullet(B/A, M) \otimes_A A' \simeq C_a^\bullet(B'/A', M').$$

- (4) For the next observation we introduce the following definition. A map of rings $A \rightarrow B$ is **faithfully flat** if B is flat over A and the map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective. One verifies easily that

- a flat map $A \rightarrow B$ is faithfully flat if and only if for every A -module M , we have $M = 0 \Leftrightarrow M \otimes B = 0$;
- a map $A \rightarrow B$ is faithfully flat if and only if for every complex of A -modules C^\bullet , we have C^\bullet is exact $\Leftrightarrow C^\bullet \otimes B$ is exact.

Then the observation is that $A \rightarrow B = \prod_{i=1}^r A[f_i^{-1}]$ is faithfully flat. Indeed, every localization is flat, so $A \rightarrow B$ is flat, and it is faithfully flat since the map

$$\text{Spec}(B) = \bigsqcup_{i=1}^r \text{Spec}(A[f_i^{-1}]) = \bigsqcup_{i=1}^r D(f_i) \longrightarrow \text{Spec}(A)$$

is surjective, as $A = (f_1, \dots, f_r)$ is equivalent to $\text{Spec}(A) = \bigcup_{i=1}^r D(f_i)$.

- (5) Finally, let $f: A \rightarrow B$ be a map of rings admitting a section $s: B \rightarrow A$, i.e. a ring homomorphism satisfying $s \circ f = \text{id}_A$. We claim that for every A -module M , the augmented Amitsur complex $C_a^\bullet(B/A, M)$ is exact. To see this, we introduce the maps

$$h^p: C_a^p(B/A; M) = \underbrace{B \otimes B \otimes \dots \otimes B}_{p+1 \text{ times}} \otimes M \longrightarrow \underbrace{B \otimes B \otimes \dots \otimes B}_{p \text{ times}} \otimes M = C_a^{p-1}(B/A; M),$$

$$b_0 \otimes \dots \otimes b_p \otimes m = b_0 \otimes \dots \otimes b_{p-1} \otimes s(b_p)m.$$

We then verify easily that on $C_a^p(B/A; M)$ we have

$$d^{p-1}h^p + h^{p+1}d^p = \text{id},$$

i.e. that h^p define a nullhomotopy. This immediately implies exactness: if $x \in \ker(d^p)$, then the above equality yields

$$x = d^{p-1}h^p(x) + h^{p+1}d^p(x) = d^{p-1}h^p(x) \in \text{im}(d^{p-1}).$$

We are now ready to state and prove the following strengthening of Lemma 9.1.1.

Lemma 9.1.2. *Let $A \rightarrow B$ be a faithfully flat map of rings and let M be an A -module. Then the augmented Amitsur complex of M*

$$C_a^\bullet(B/A, M) = \left[0 \longrightarrow M \longrightarrow B \otimes M \longrightarrow B \otimes B \otimes M \longrightarrow \dots \right]$$

is exact.

Proof. Since $A \rightarrow B$ is faithfully flat, it is enough to check that $C_a^\bullet(B/A, M) \otimes_A B$ is exact. By Observation (3), we have

$$C_a^\bullet(B/A, M) \otimes_A B \simeq C_a^\bullet(B \otimes_A B/B, B \otimes_A M).$$

The map $B \rightarrow B \otimes_A B$ admits a “tautological” (or “diagonal”) section $s: B \otimes_A B \rightarrow B$ given by $s(b \otimes b') = bb'$. Thus Observation (5) implies that $C_a^\bullet(B \otimes_A B/B, B \otimes_A M)$ is exact. \square

Remarks 9.1.3. 1. The word “augmented” refers to starting the complex at -1 . The non-augmented version is

$$C^\bullet(B/A, M) = \left[B \otimes M \longrightarrow B \otimes B \otimes M \longrightarrow \cdots \right],$$

which is then a resolution of M .

2. Just as the exactness of the Čech complex in degrees $-1, 0$ allows us to construct the structure sheaf \mathcal{O}_X on $X = \text{Spec}(A)$ by associating $A[f^{-1}]$ to $D(f)$, the variant with an A -module M allows us to construct a sheaf of \mathcal{O}_X -modules \tilde{M} on \mathcal{O}_X with $\tilde{M}(D(f)) = M[f^{-1}]$. Such sheaves of modules on X are called **quasi-coherent** and will be discussed in greater detail soon.
3. The exactness of the entire Čech complex, not just its beginning, will be used later to show that for a quasi-coherent sheaf \mathcal{F} on an affine scheme X we have the vanishing of sheaf cohomology

$$H^p(X, \mathcal{F}) = 0 \quad \text{for all } p > 0.$$

4. The generality offered by Lemma 9.1.2 can be useful in many situations. Examples of faithfully flat maps are given by surjective étale maps, which are used to build the étale topology. Another (though related) example relates this complex to Galois cohomology:

Let L/K be a finite Galois field extension and let $G = \text{Gal}(L/K)$ be its Galois group. Then the map

$$L \otimes_K L \longrightarrow \prod_{\sigma \in G} L, \quad x_0 \otimes x_1 \mapsto (x_0 \sigma(x_1))_{\sigma \in G}$$

is an isomorphism of L -algebras. More generally, we have isomorphisms

$$\underbrace{L \otimes_K L \otimes_K \cdots \otimes_K L}_{p+1} \simeq \prod_{\sigma_1, \dots, \sigma_p \in G} L, \quad x_0 \otimes \cdots \otimes x_p \mapsto x_0 \cdot \sigma_1(x_1) \cdot (\sigma_1 \sigma_2)(x_2) \cdot \cdots \cdot (\sigma_1 \cdots \sigma_p)(x_p).$$

Thus the Amitsur complex for $A = M = K$ and $B = L$ can be written in the form

$$0 \longrightarrow K \longrightarrow L \longrightarrow L^G \longrightarrow L^{G \times G} \longrightarrow \cdots$$

which one can verify to be the (augmented) bar complex computing the group cohomology $H^*(G, L)$. We deduce Hilbert’s Theorem 90:

$$H^p(G, L) = \begin{cases} K & p = 0, \\ 0 & p > 0. \end{cases}$$

9.2. Maps to affine schemes

We give a proof of the following result stated last time.

Proposition 9.2.1. *Let X be an affine scheme and let Y be a locally ringed space. Then the map*

$$\phi \mapsto \phi^*: \text{Hom}_{\text{LRS}}(Y, X) \longrightarrow \text{Hom}_{\text{Rings}}(\mathcal{O}(X), \mathcal{O}(Y))$$

is bijective.

Proof. Let $A = \mathcal{O}(X)$ so that $X = \text{Spec}(A)$. We first show **injectivity**. Let $\phi, \psi: Y \rightarrow X$ be maps of locally ringed spaces such that $\phi^* = \psi^*: A \rightarrow \mathcal{O}(Y)$. We first show that $\phi(y) = \psi(y)$ for every $y \in Y$. Suppose otherwise: $\phi(y) = x \neq x' = \psi(y)$. Thus the prime ideals corresponding to $x, x' \in X = \text{Spec}(A)$ are distinct, and hence (up to swapping x and x') there exists $f \in A$ such that $f \in \mathfrak{p}_x \setminus \mathfrak{p}_{x'}$. Since $\mathcal{O}_{X,x} = A_{\mathfrak{p}_x}$

and $\mathcal{O}_{X,x'} = A_{\mathfrak{p}_{x'}}$, this means that $f(x) = 0$ and $f(x') \neq 0$. Since ϕ and ψ are maps of locally ringed spaces, the homomorphisms

$$\phi^*: A_{\mathfrak{p}_x} = \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y} \quad \text{and} \quad \psi^*: A_{\mathfrak{p}_{x'}} = \mathcal{O}_{X,x'} \rightarrow \mathcal{O}_{Y,y}$$

are local, and we conclude that $\phi^*(f)(y) = 0$ and $\psi^*(f)(y) \neq 0$, contradicting $\phi^*(f) = \psi^*(f)$.

Next, we need to show that for every open $U \subseteq X$, the two maps

$$\phi^*, \psi^*: \mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(\phi^{-1}(U)) = \mathcal{O}_Y(\psi^{-1}(U))$$

are equal. It suffices to check this for a basic open $U = D(f)$ for some $f \in A$. In this case we have

$$\phi^{-1}(U) = \psi^{-1}(U) = D(g), \quad g = \phi^*(f) = \psi^*(f)$$

Both maps in question are the dotted arrow making the square below commute.

$$\begin{array}{ccc} A = \mathcal{O}_X(X) & \xrightarrow{\phi^* = \psi^*} & \mathcal{O}_Y(Y) \\ \downarrow & & \downarrow \\ A[f^{-1}] = \mathcal{O}_X(D(f)) & \cdots \cdots \cdots \rightarrow & \mathcal{O}_Y(D(g)) \end{array}$$

and hence they are equal (by universal property of the localization $A[f^{-1}]$).

We turn to proving **surjectivity**. Let $\phi^*: A \rightarrow \mathcal{O}(Y)$ be a ring homomorphism for which we seek to construct the corresponding map $Y \rightarrow \text{Spec}(A)$. We first construct the map of sets: for $y \in Y$, let $\phi(y) \in X = \text{Spec}(A)$ be the point corresponding to the prime ideal

$$\mathfrak{p} = \ker \left(A \xrightarrow{\phi^*} \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \kappa(y) \right) = \{f \in A : \phi^*(f)(y) = 0\}.$$

We verify easily that $\phi^{-1}(D(f)) = D(\phi^*(f))$. We have shown previously that this is an open subset. Since the $D(f)$ form a base of the topology on $\text{Spec}(A)$, this shows that ϕ is continuous.

In order to upgrade this to a map of locally ringed spaces, it suffices to construct compatible maps, for every $f \in A$,

$$\phi^*: A[f^{-1}] = \mathcal{O}_X(D(f)) \longrightarrow \mathcal{O}_Y(\phi^{-1}(D(f))) = \mathcal{O}_Y(D(\phi^*f)).$$

We have shown earlier that ϕ^*f is a unit in $\mathcal{O}_Y(D(\phi^*f))$. Thus (again, universal property of localization) there exists a unique A -algebra map above. We omit the easy verification that this gives a map of ringed spaces.

Finally, for $y \in Y$ and $x = \phi(y)$, the induced map on stalks is

$$A_{\mathfrak{p}_x} = \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$$

which is local by our definition of the prime ideal \mathfrak{p}_x . Thus ϕ is a map of locally ringed spaces inducing ϕ^* on global sections of structure sheaves. \square

9.3. Properties of schemes and morphisms of schemes

Today, we just introduce the vocabulary.

Definition 9.3.1. Let X be a scheme.

- (a) We denote by $|X|$ the underlying topological space of X . We say that X is **connected**, **quasi-compact**, or **irreducible** if $|X|$ has this property.

- (b) We say that X is **reduced** if $\mathcal{O}_X(U)$ is reduced for every open $U \subseteq X$, and that X is **integral** if it is reduced and irreducible (equivalently, if $\mathcal{O}_X(U)$ is a domain for every non-empty open $U \subseteq X$).
- (c) We say that X is **noetherian** if it admits a finite open cover $X = U_1 \cup \dots \cup U_r$ with $U_i \simeq \text{Spec}(A_i)$ where the rings A_1, \dots, A_r are noetherian.

For a ring A , by a **scheme over A** we mean a scheme X endowed with a map $X \rightarrow \text{Spec}(A)$, or equivalently (by Proposition 9.2.1) one for which $\mathcal{O}_X(U)$ is an A -algebra for every $U \subseteq X$ (and such that the restriction maps are homomorphisms of A -algebras).

Definition 9.3.2. Let $f: Y \rightarrow X$ be a morphism of schemes. We say that f is

- (a) **quasi-compact** if for every quasi-compact open $U \subseteq X$, the preimage $f^{-1}(U) \subseteq Y$ is quasi-compact;
- (b) **affine** if for every affine open $U \subseteq X$, the preimage $f^{-1}(U) \subseteq Y$ is affine;
- (c) an **open immersion** if it induces an isomorphism from Y onto an open subscheme of X ;
- (d) a **closed immersion** if it induces a homeomorphism from Y onto a closed subset of X and the map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is surjective;
- (e) **finite** if it is affine and for every affine open $U \subseteq X$, the map $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(f^{-1}(U))$ is finite;
- (f) **locally of finite type** if it is locally of the form $\text{Spec}(B) \rightarrow \text{Spec}(A)$ for a map of rings $A \rightarrow B$ such that B is a finitely generated A -algebra (more precisely: for every $y \in Y$ there exist affine open neighborhoods $y \in V$ and $f(y) \in U$ such that $f(V) \subseteq U$ and the map $f^*: \mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(V)$ is of finite type);
- (g) of **finite type** if it is locally of finite type and quasi-compact.

We issue the following warning. While an open immersion is uniquely determined by its image, there may be multiple closed immersions $Y \rightarrow X$ whose image is the same closed subset $Z \subseteq X$. For example, if $X = \text{Spec}(A)$, then closed immersions $Y \rightarrow X$ are of the form $\text{Spec}(A/I) \rightarrow \text{Spec}(A)$, and the corresponding closed subset is $Z = V(I)$. Thus knowing Z we only know the radical \sqrt{I} .

9.4. Fiber products of schemes

Proposition 9.4.1. Let $f: X \rightarrow S$ and $g: Y \rightarrow S$ be maps of schemes.

- (a) The fiber product $X \times_S Y$ exists in the category of schemes.
- (b) If X, Y , and S are affine then so is $X \times_S Y$, more precisely we have

$$X \times_S Y \simeq \text{Spec}(\mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(Y)).$$

- (c) In general, if $S = \bigcup_{\alpha \in I} S_\alpha$ is an affine open cover, and if $f^{-1}(S_\alpha) = \bigcup_{\beta} X_{\alpha\beta}$, $g^{-1}(S_\alpha) = \bigcup_{\gamma} Y_{\alpha\gamma}$ are affine open covers, then $X \times_S Y$ has an affine open cover by

$$X_{\alpha\beta} \times_{S_\alpha} Y_{\alpha\gamma} = \text{Spec}(\mathcal{O}(X_{\alpha\beta}) \otimes_{\mathcal{O}(S_\alpha)} \mathcal{O}(Y_{\alpha\gamma})).$$

- (d) If $Y = \text{Spec}(\kappa(s))$ for some $s \in S$, then $X \times_S Y = f^{-1}(s)$ as a topological space.

Proof. We first show (b), which is a direct application of Proposition 9.2.1 and the fact that $\mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(Y)$ is the pushout of the two maps $f^*: \mathcal{O}(S) \rightarrow \mathcal{O}(X)$ and $g^*: \mathcal{O}(S) \rightarrow \mathcal{O}(Y)$ in the category of rings. For any scheme Z we have

$$\begin{aligned} \mathrm{Hom}(Z, \mathrm{Spec}(\mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(Y))) &= \mathrm{Hom}(\mathcal{O}(X) \otimes_{\mathcal{O}(S)} \mathcal{O}(Y), \mathcal{O}(Z)) \\ &= \mathrm{Hom}(\mathcal{O}(X), \mathcal{O}(Z)) \times_{\mathrm{Hom}(\mathcal{O}(S), \mathcal{O}(Z))} \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(Z)) \\ &= \mathrm{Hom}(Z, X) \times_{\mathrm{Hom}(Z, S)} \mathrm{Hom}(Z, Y). \end{aligned}$$

Parts (c) and (a) are shown together using (b) in a slightly tedious fashion, see [Har77, Theorem II 3.3]. For (c), we may assume that S and X are affine, and then the result follows from the corresponding fact about rings (proved in the commutative algebra course). \square

The following lemma is related to the fact that affine schemes are “separated” (a notion we only introduced for algebraic sets in Lecture 3).

Lemma 9.4.2. *Let X be an affine scheme and let $U, V \subseteq X$ be two affine open subsets. Then $U \cap V$ is affine.*

Proof. Trivial observation: a map $Z \rightarrow X$ from a scheme Z factors through the inclusion of an open subset $U \rightarrow X$ if and only if the image of Z in X is contained in U ; such a factorization is unique if it exists. From this, we see that $U \cap V$ is isomorphic to the fiber product $U \times_X V$: a map $Z \rightarrow X$ factors through $U \cap V$ if and only if the image of Z in X is contained in both U and V . But then $U \times_X V$ is affine by Proposition 9.4.1(b). \square

9.5. Schemes versus algebraic sets

We shall use the following result to go back and forth between schemes and algebraic sets.

Theorem 9.5.1. *Let k be an algebraically closed field.*

(a) *Let X be a reduced scheme of finite type over k . Let $X(k)$ be its set of closed points (which coincides with $\mathrm{Hom}_k(\mathrm{Spec}(k), X)$ by Nullstellensatz), endowed with the induced topology. For an open $U \subseteq X(k)$, there exists a unique open $U' \subseteq X$ with $U' \cap X(k) = U$. The association*

$$U \mapsto \mathcal{O}(U) = \mathcal{O}_X(U')$$

defines a sheaf of rings on $X(k)$. The space with functions $(X(k), \mathcal{O})$ is an algebraic set.

(b) *This construction defines an equivalence of categories*

$$\{\text{reduced schemes of finite type over } k\} \xrightarrow{\sim} \{\text{algebraic sets over } k\}.$$

10. Lecture 10 (Feb 24): Sheaves of modules

Our treatment of quasi-coherent and coherent sheaves largely follows Hartshorne II.5, except that we replace some of the hands-on computations in the proof with the slightly more conceptual arguments using the exactness of the Čech complex proved in Lecture 9. (Our definition of the sheaves $\mathcal{O}(d)$ in \mathbb{P}^n is also different, since we avoid discussing the link between coherent sheaves on \mathbb{P}^n and graded modules over the polynomial ring.)

Recommended reading: Kempf §5, Hartshorne II.5

10.1. Sheaves of modules

Definition 10.1.1. Let \mathcal{O} be a sheaf of (commutative, unital) rings on a topological space X . An \mathcal{O} -**module** is a sheaf of abelian groups \mathcal{M} together with a map of sheaves

$$\mathcal{O} \times \mathcal{M} \longrightarrow \mathcal{M}$$

giving $\mathcal{M}(U)$ the structure of an $\mathcal{O}(U)$ -module for every open $U \subseteq X$. A morphism of \mathcal{O} -modules is a map of sheaves $f: \mathcal{F} \rightarrow \mathcal{G}$ such that $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}(U)$ -modules for every open $U \subseteq X$.

Remarks 10.1.2. (a) Equivalent characterizations of the structure of an \mathcal{O} -module on a sheaf of abelian groups \mathcal{M} :

- for every $U \subseteq X$, the structure of an $\mathcal{O}(U)$ -module on $\mathcal{M}(U)$ such that the restriction maps $\mathcal{M}(U) \rightarrow \mathcal{M}(V)$ are maps of $\mathcal{O}(U)$ -modules; or
- a homomorphism of sheaves of rings $\mathcal{O} \rightarrow \mathcal{H}om(\mathcal{M}, \mathcal{M})$.

(b) Many sources call an \mathcal{O} -module a “sheaf of \mathcal{O} -modules,” though that is a bit redundant as there is no other possible meaning of “ \mathcal{O} -module.”

(c) Familiar notions from commutative algebra have natural \mathcal{O} -module analogs. Kernels, cokernels, images work in the obvious way. Tensor product is identified as the sheafification of the obvious tensor product of presheaves $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{M}'(U)$. Further examples: an ideal is an \mathcal{O} -submodule $\mathcal{I} \subseteq \mathcal{O}$, and an \mathcal{O} -algebra is a map $\mathcal{O} \rightarrow \mathcal{A}$ of sheaves of rings, or equivalently an \mathcal{O} -module \mathcal{A} equipped with a map $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \rightarrow \mathcal{A}$ satisfying suitable axioms. For a map of sheaves of rings $\mathcal{O} \rightarrow \mathcal{O}'$, there is the usual adjunction between the forgetful functor $\mathbf{Mod}_{\mathcal{O}'} \rightarrow \mathbf{Mod}_{\mathcal{O}}$ (right adjoint) and the tensor product $(-) \otimes_{\mathcal{O}} \mathcal{O}': \mathbf{Mod}_{\mathcal{O}} \rightarrow \mathbf{Mod}_{\mathcal{O}'}$ (left adjoint).

(d) If it is understood from the context that we work with \mathcal{O}_X -modules on a ringed space X , we write $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ (resp. $\mathcal{H}om(\mathcal{F}, \mathcal{G})$) for the set (resp. sheaf) of all \mathcal{O}_X -module maps $\mathcal{F} \rightarrow \mathcal{G}$.

(e) Let $\phi: Y \rightarrow X$ be a map of ringed spaces. If \mathcal{G} is a \mathcal{O}_Y -module, then $\phi_*\mathcal{G}$ is a $\phi_*\mathcal{O}_Y$ -module, and hence can be regarded as a \mathcal{O}_X -module thanks to the given map $\phi^*: \mathcal{O}_X \rightarrow \phi_*\mathcal{O}_Y$. There is also a module pull-back functor $\mathcal{F} \mapsto \phi^*\mathcal{F}$ from \mathcal{O}_X -modules to \mathcal{O}_Y -modules, defined as follows

$$\phi^*\mathcal{F} = \phi_{\mathrm{Ab}}^*(\mathcal{F}) \otimes_{\phi_{\mathrm{Ab}}^*(\mathcal{O}_X)} \mathcal{O}_Y$$

where ϕ_{Ab}^* denotes pull-back of sheaves abelian groups³ (see Lecture 7). In other words, introducing the sheaf of rings $\mathcal{O}_{Y/X} = \phi_{\mathrm{Ab}}^*(\mathcal{O}_X)$, the functor is the composition of two functors

$$\mathbf{Mod}_{\mathcal{O}_X} \xrightarrow{\phi_{\mathrm{Ab}}^*} \mathbf{Mod}_{\mathcal{O}_{Y/X}} \xrightarrow{(-) \otimes_{\mathcal{O}_{Y/X}} \mathcal{O}_Y} \mathbf{Mod}_{\mathcal{O}_X}.$$

It is easy to deduce from this presentation that ϕ^* is left adjoint to ϕ_* .

³Certain sources (e.g. Hartshorne) use the notation $\phi^{-1}(\mathcal{F})$ for our $\phi_{\mathrm{Ab}}^*(\mathcal{F})$.

Examples 10.1.3. 1. If X is a C^∞ -manifold and \mathcal{O} is the sheaf of smooth real-valued functions on X , the sheaf \mathcal{M} of differential 1-forms on X has a natural structure of an \mathcal{O} -module. This simply says that we can multiply a differential form by a smooth function.

2. Let $X = \text{Spec}(A)$ be an affine scheme and let M be an A -module. As we shall see directly below, we can turn M into a \mathcal{O}_X -module \tilde{M} whose value on a distinguished open $D(f) \subseteq X$ is $M[f^{-1}]$.

Lemma 10.1.4 (compare Corollary 8.2.2). *Let X be either an affine scheme or an affine algebraic set, let $A = \mathcal{O}(X)$, and let M be an A -module. There exists a unique \mathcal{O}_X -module \tilde{M} such that for every $f \in A$ we have*

$$\tilde{M}(D(f)) = M[f^{-1}]$$

(and such that for $f, g \in A$, the restriction map $\tilde{M}(D(f)) \rightarrow \tilde{M}(D(fg))$ coincides with the natural map $A[f^{-1}] \rightarrow A[(fg)^{-1}]$). Moreover, for every $x \in X$ corresponding to a prime ideal $\mathfrak{p}_x \subseteq A$, we have

$$\tilde{M}_x \simeq M_{\mathfrak{p}_x}.$$

Proof. Same as the proof of Corollary 8.2.2, only now we use Lemma 9.1.1 in place of Lemma 8.2.1. \square

Remark 10.1.5. A shorter construction of the sheaf \tilde{M} (which works in more general settings). Consider the one-point ringed space $\star_A = (\star, A)$ with structure sheaf given by A . There is an obvious map $p: X \rightarrow \star_A$. On the other hand, A -modules are \mathcal{O} -modules of \star_A , and for an A -module M , we have $\tilde{M} = p^*M$.

The above lemma constructs a functor

$$\widetilde{(-)}: \mathbf{Mod}_{\mathcal{O}(X)} \rightarrow \mathbf{Mod}_{\mathcal{O}_X}.$$

We shall study its properties in the next few lemmas.

Lemma 10.1.6 (Hartshorne II Proposition 5.2(d,e)). *Let $\phi: Y \rightarrow X$ be a map of affine schemes or affine algebraic sets, and let $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$.*

(a) *Let M be an A -module. Then $f^*(\tilde{M})$ is naturally isomorphic to $\widetilde{M \otimes_A B}$. In other words, the diagram of functors*

$$\begin{array}{ccc} \mathbf{Mod}_A & \xrightarrow{(-) \otimes_A B} & \mathbf{Mod}_B \\ \downarrow & & \downarrow \\ \mathbf{Mod}_{\mathcal{O}_X} & \xrightarrow{\phi^*} & \mathbf{Mod}_{\mathcal{O}_Y} \end{array}$$

commutes.

(b) *Let N be an A -module. Then $f_*(\tilde{N})$ is naturally isomorphic to \tilde{N} for N treated as an A -module. In other words, the diagram of functors*

$$\begin{array}{ccc} \mathbf{Mod}_A & \xleftarrow{\text{forget}} & \mathbf{Mod}_B \\ \downarrow & & \downarrow \\ \mathbf{Mod}_{\mathcal{O}_X} & \xleftarrow{\phi_*} & \mathbf{Mod}_{\mathcal{O}_Y} \end{array}$$

commutes.

Proof. This is clear once you unwrap the definitions. \square

Lemma 10.1.7 (Hartshorne II Proposition 5.2(a,b,c)). *Let $X = \text{Spec}(A)$. The functor $M \mapsto \tilde{M}$ from A -modules to \mathcal{O}_X -modules is fully faithful, conservative, and exact, and commutes with tensor product and arbitrary direct sums. For every \mathcal{O}_X -module \mathcal{F} we have*

$$\text{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) = \text{Hom}_A(M, \mathcal{F}(X)).$$

Proof. Since $\tilde{M}(X) = M$, the functor is faithful and conservative. Let $f: \tilde{M} \rightarrow \tilde{N}$ be a map, and let $f': \tilde{M} \rightarrow \tilde{N}$ be the map induced by the map $M \rightarrow N$ obtained by applying global sections to f . At every point $x \in X$, the stalks of f and f' agree, and hence $f = f'$, which shows fullness. A similar argument involving stalks proves exactness. The assertions about \otimes and \oplus are clear from the definition of \tilde{M} . The final assertion is easiest to see through the lens of Remark 10.1.5. \square

10.2. Quasi-coherent sheaves

Definition 10.2.1. Let X be either a scheme or an algebraic set. A \mathcal{O}_X -module \mathcal{F} is **quasi-coherent** if there exists an affine open cover $X = \bigcup U_\alpha$ such that $\mathcal{F}|_{U_\alpha}$ is of the form \tilde{M}_α for some $\mathcal{O}(U_\alpha)$ -module M_α . If X is either a locally noetherian scheme or an algebraic set, we say that \mathcal{F} is **coherent** if the same condition holds with each M_α a finitely generated $\mathcal{O}(U_\alpha)$ -module.

Example 10.2.2 (A non-quasi-coherent \mathcal{O}_X -module). Consider the open immersion

$$j: U = \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathbb{A}^1 = X.$$

The extension by zero $j_! \mathcal{O}_U \subseteq \mathcal{O}_X$ is defined as

$$(j_! \mathcal{O}_U)(V) = \{f \in \mathcal{O}_U(U \cap V) : f_x = 0 \text{ for all } x \in V \setminus U\}$$

(see [Hartshorne Ex. II 1.19(b)]). In our case, this means that

$$(j_! \mathcal{O}_U)(V) = \begin{cases} \mathcal{O}(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise.} \end{cases}$$

Then $j_!(\mathcal{O}_U)(U) \neq 0$, so $j_!(\mathcal{O}_U) \neq 0$. On the other hand, we have $j_!(\mathcal{O}_U)(X) = 0$. It follows that $j_! \mathcal{O}_U$ is not quasi-coherent (since it is nonzero but has no global sections). The quotient $\mathcal{O}_X / j_! \mathcal{O}_U$ is the skyscraper at 0 with value $\mathcal{O}_{X,0}$, which is not quasi-coherent either. See also [Har77, Example II 5.2.3] and [Mum99, §III.1, Example A, p. 142].

Remark 10.2.3. Quasi-coherent and coherent sheaves have the following simple characterization, which makes sense on any ringed space⁴. An \mathcal{O}_X -module is quasi-coherent if and only if every $x \in X$ has an open neighborhood $U \subseteq X$ on which \mathcal{F} admits a presentation of the form

$$\mathcal{O}_U^{\oplus J} \longrightarrow \mathcal{O}_U^{\oplus I} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

for some sets I and J . Moreover, \mathcal{F} is coherent if and only if the same holds with I and J finite. It follows that coherent sheaves are the smallest class of \mathcal{O}_U -modules on opens $U \subseteq X$ such that (a) being coherent is a local condition, (b) \mathcal{O}_U is coherent, (c) coherent sheaves are closed under direct sums and cokernels.

Lemma 10.2.4. *Let $X = \text{Spec}(A)$ be an affine scheme and let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then the natural map*

$$\widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F}$$

(the counit of the adjunction in Lemma 10.1.7) is an isomorphism. In particular, \mathcal{F} is of the form \tilde{M} .

⁴However, the resulting notion will not always be so well-behaved in such generality.

Proof. Note that since the functor $M \rightarrow \tilde{M}$ is fully faithful and the global sections functor is its right adjoint (Lemma 10.1.7), an \mathcal{O}_X -module is of the form \tilde{M} if and only if its counit map $\widetilde{\mathcal{F}(X)} \rightarrow \mathcal{F}$ is an isomorphism. For brevity, let us call such \mathcal{O}_X -modules *in the essential image* (of the functor $M \mapsto \tilde{M}$). By Lemma 10.1.7, being in the essential image is closed under kernels, cokernels, and finite direct sums.

By definition of a quasi-coherent \mathcal{O}_X -module, there exists an affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that $\mathcal{F}|_{U_\alpha}$ is of the form \tilde{M}_α for some $\mathcal{O}(U_\alpha)$ -module M_α . Since $\text{Spec}(A)$ is quasi-compact, we may assume that the index set I is finite. By Lemma 10.1.6, if $j_\alpha: U_\alpha \rightarrow X$ is the inclusion, the \mathcal{O}_X -module

$$\mathcal{F}_\alpha := j_{\alpha,*}j_\alpha^*(\mathcal{F}) = j_{\alpha,*}(\mathcal{F}|_{U_\alpha}) = j_{\alpha,*}(\tilde{M}_\alpha)$$

is equal to \tilde{M}_α where M_α is treated as an A -module (via the forgetful functor $\mathbf{Mod}_{\mathcal{O}(U_\alpha)} \rightarrow \mathbf{Mod}_A$), and hence in the essential image. In particular the sheaf

$$\mathcal{F}_0 = \bigoplus_{\alpha \in I} \mathcal{F}_\alpha = \bigoplus_{\alpha \in I} j_{\alpha,*}j_\alpha^*(\mathcal{F})$$

is in the essential image (as this condition is closed under finite direct sums). Since the intersections $U_\alpha \cap U_\beta$ are affine (Lemma 9.4.2), we can apply the same reasoning to the sheaf

$$\mathcal{F}_1 = \bigoplus_{\alpha, \beta \in I} j_{\alpha\beta,*}j_{\alpha\beta}^*(\mathcal{F}), \quad j_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow X.$$

Now, by the sheaf condition for \mathcal{F} , the following sequence of \mathcal{O}_X -modules is exact

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1, \quad d(s)_{\alpha\beta} = s_\alpha - s_\beta.$$

and hence $\mathcal{F} = \ker(d: \mathcal{F}_0 \rightarrow \mathcal{F}_1)$ is in the essential image. □

11. Lecture 11 (Feb 26): Quasi-coherent sheaves

Recommended reading: Kempf §5, Hartshorne II.5

11.1. More on quasi-coherent sheaves

Lemma 11.1.1. *Let $X = \text{Spec}(A)$ be an affine scheme and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of \mathcal{O}_X -modules. If \mathcal{F}' is quasi-coherent, then

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact.

Proof. As sequence is always left-exact, we have to show that the right map is surjective. Let $s \in \mathcal{F}''(X)$. Since $\mathcal{F} \rightarrow \mathcal{F}''$ is surjective, there exists an open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that $s|_{U_\alpha}$ lies in the image of $\mathcal{F}(U_\alpha) \rightarrow \mathcal{F}''(U_\alpha)$ for every $\alpha \in I$. Since X is quasi-compact and distinguished affine opens form a base, we may assume that the index set I is finite and that $U_\alpha = D(f_\alpha)$ for some elements $f_\alpha \in A$ generating the unit ideal. Refining the cover further, we may assume that in addition the restriction of \mathcal{F}' to each U_α is of the form \tilde{M}_α for some $\mathcal{O}(U_\alpha)$ -module M_α . Now consider the commutative diagram of Čech complexes:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{F}'(X) & \longrightarrow & \prod_{\alpha \in I} \mathcal{F}'(U_\alpha) & \longrightarrow & \prod_{\alpha, \beta \in I} \mathcal{F}'(U_\alpha \cap U_\beta) & \longrightarrow & \prod_{\alpha, \beta, \gamma \in I} \mathcal{F}'(U_\alpha \cap U_\beta \cap U_\gamma) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \prod_{\alpha \in I} \mathcal{F}(U_\alpha) & \longrightarrow & \prod_{\alpha, \beta \in I} \mathcal{F}(U_\alpha \cap U_\beta) & \longrightarrow & \prod_{\alpha, \beta, \gamma \in I} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{F}''(X) & \longrightarrow & \prod_{\alpha \in I} \mathcal{F}''(U_\alpha) & \longrightarrow & \prod_{\alpha, \beta \in I} \mathcal{F}''(U_\alpha \cap U_\beta) & \longrightarrow & \prod_{\alpha, \beta, \gamma \in I} \mathcal{F}''(U_\alpha \cap U_\beta \cap U_\gamma) & \longrightarrow & \dots
 \end{array}$$

We make the following observations:

- The diagram commutes (since the formation of the Čech complex is functorial).
- The columns are exact (by left-exactness of taking sections), i.e. exact after forgetting the bottom zero.
- All three rows are exact until the third column (the one with $\prod_{\alpha, \beta \in I}$). This is the sheaf condition.
- By Lemma 10.2.4 we know that \mathcal{F}' is of the form \tilde{M} for some A -module M , and hence the top row (the one with \mathcal{F}') is exact, by Lemma 9.1.1 proved earlier.

To finish the argument, we perform a tedious diagram chase which is best explained on the blackboard or done by oneself. We shall use the symbol d for the horizontal (Čech) differentials. By assumption, the image $(s|_{U_\alpha}) \in \prod_{\alpha \in I} \mathcal{F}''(U_\alpha)$ of $s \in \mathcal{F}''(X)$ is the image of an element $(t_\alpha) \in \prod_{\alpha \in I} \mathcal{F}(U_\alpha)$. If $d(t_\alpha) = 0 \in \prod_{\alpha, \beta \in I} \mathcal{F}(U_\alpha \cap U_\beta)$, then by exactness of the middle row it comes from a unique $t \in \mathcal{F}(X)$, which then maps to $s \in \mathcal{F}''(X)$ and we are done. In general, the image

$$d(t_\alpha) = (t_\alpha|_{U_\alpha \cap U_\beta} - t_\beta|_{U_\alpha \cap U_\beta}) \in \prod_{\alpha, \beta \in I} \mathcal{F}(U_\alpha \cap U_\beta)$$

maps via the vertical map to

$$d(s_\alpha) = (s|_{U_\alpha \cap U_\beta} - s|_{U_\alpha \cap U_\beta}) = 0 \in \prod_{\alpha, \beta \in I} \mathcal{F}''(U_\alpha \cap U_\beta),$$

and hence by exactness of the third column it is the image of an element $(u_{\alpha\beta}) \in \prod_{\alpha, \beta \in I} \mathcal{F}'(U_\alpha \cap U_\beta)$. Again, the image of $d(u_{\alpha\beta})$ in $\prod_{\alpha, \beta, \gamma \in I} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma)$ is $d(d(t_\alpha)) = 0$. By the injectivity of the vertical maps from the 1st to the 2nd row, we deduce that $d(u_{\alpha\beta}) = 0$. By exactness of the 1st row (here we use the additional assumption $\mathcal{F}' = \tilde{M}$) we obtain a $(u_\alpha) \in \prod_{\alpha \in I} \mathcal{F}'(U_\alpha)$ such that

$$(u_\alpha|_{U_\alpha \cap U_\beta} - u_\beta|_{U_\alpha \cap U_\beta}) = d(u_\alpha) = (u_{\alpha\beta}).$$

Now set

$$t'_\alpha = t_\alpha - u_\alpha.$$

As u_α maps to zero in $\mathcal{F}''(U_\alpha)$, the image of (t'_α) in $\prod_{\alpha \in I} \mathcal{F}''(U_\alpha)$ equals (s_α) . We check that

$$d(t'_\alpha) = d(t_\alpha) - d(u_\alpha) = d(t_\alpha) - (u_{\alpha\beta}) = 0,$$

and again we deduce the existence of a unique $t' \in \mathcal{F}(X)$ whose image in $\mathcal{F}''(X)$ equals s . \square

Corollary 11.1.2. *Let X be a scheme or an algebraic set and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of \mathcal{O}_X -modules. If \mathcal{F}' and \mathcal{F}'' are quasi-coherent, then so is \mathcal{F} .

Proof. The assertion is local so we may assume that X is affine. By Lemma 11.1.1, the sequence of $\mathcal{O}(X)$ -modules

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{\mathcal{F}'(X)} & \longrightarrow & \widetilde{\mathcal{F}(X)} & \longrightarrow & \widetilde{\mathcal{F}''(X)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' \longrightarrow 0 \end{array}$$

in which the vertical arrows are the counit maps (see Lemma 10.2.4). The top row is exact by exactness of the functor $\widetilde{(-)}$. Since \mathcal{F}' and \mathcal{F}'' are quasi-coherent, the left and right maps are isomorphisms. The five lemma implies that so is the middle one, so that \mathcal{F} is quasi-coherent. \square

Functorial properties of quasi-coherent sheaves:

Lemma 11.1.3. *Let $f: Y \rightarrow X$ be a map of schemes or algebraic sets.*

- (a) *For every quasi-coherent \mathcal{O}_X -module \mathcal{F} , the pull-back $f^*\mathcal{F}$ is quasi-coherent.*
- (b) *The same holds for coherent sheaves if X and Y are algebraic sets or locally noetherian schemes.*
- (c) *Suppose that X and Y are noetherian schemes⁵ or algebraic sets. Then for every quasi-coherent \mathcal{O}_Y -module \mathcal{G} , the push-forward $f_*\mathcal{G}$ is a quasi-coherent \mathcal{O}_X -module.*

⁵More generally the proof shows the claim if f is quasi-compact and quasi-separated [SP Section 01KH].

Proof. Assertions (a) and (b) are local on Y , and hence are easily reduced to the case X and Y affine, in which situation we might apply Lemma 10.1.6(a). For (c), this is local on X , and we may assume that X is affine. If Y is also affine, we win by Lemma 10.1.6(b). In general, since Y is quasi-compact, we can cover it by finitely many affines U_α ($\alpha = 1, \dots, n$). Since Y is a noetherian topological space, the intersections $U_\alpha \cap U_\beta$ are quasi-compact, so we can cover them by finitely many affines $U_{\alpha\beta\gamma}$ ($\gamma = 1, \dots, m(\alpha, \beta)$). Let $j_\alpha: U_\alpha \rightarrow X$ and $j_{\alpha\beta\gamma}: U_{\alpha\beta\gamma} \rightarrow X$ be the inclusions. We can now express \mathcal{F} as the kernel

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_\alpha j_{\alpha,*} j_\alpha^* \mathcal{F} \longrightarrow \bigoplus_{\alpha,\beta,\gamma} j_{\alpha\beta\gamma,*} j_{\alpha\beta\gamma}^* \mathcal{F},$$

(technically these are more canonically the products not direct sums, but since the index sets are finite, it does not matter). Since f_* is left-exact and commutes with direct sums, we obtain a description of $f_*\mathcal{F}$ as the kernel

$$\begin{array}{ccccc} 0 & \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_* \bigoplus_\alpha j_{\alpha,*} j_\alpha^* \mathcal{F} & \longrightarrow & f_* \bigoplus_{\alpha,\beta,\gamma} j_{\alpha\beta\gamma,*} j_{\alpha\beta\gamma}^* \mathcal{F} \\ & & & & \parallel & & \parallel \\ & & & & \bigoplus_\alpha (f \circ j_\alpha)_*(\mathcal{F}|_{U_\alpha}) & \longrightarrow & \bigoplus_{\alpha\beta\gamma} (f \circ j_{\alpha\beta\gamma})_*(\mathcal{F}|_{U_{\alpha\beta\gamma}}). \end{array}$$

Since each $f \circ j_\alpha$ and $f \circ j_{\alpha\beta\gamma}$ is a map between affines, every summand above is quasi-coherent, and hence so is \mathcal{F} . \square

11.2. Fibers of coherent sheaves

Definition 11.2.1. Let X be a scheme or an algebraic set and let \mathcal{F} be an \mathcal{O}_X -module. The **fiber** of \mathcal{F} at a point $x \in X$ is

$$\mathcal{F}(x) \in \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x),$$

a vector space over the residue field $\kappa(x)$.

As a corollary of Nakayama's lemma, we have:

Lemma 11.2.2. *Let X be a locally noetherian scheme or an algebraic set and let \mathcal{F} be a coherent \mathcal{O}_X -module. Then:*

- (a) $\dim_{\kappa(x)} \mathcal{F}(x) < \infty$ for all $x \in X$;
- (b) a map of quasi-coherent sheaves $\mathcal{F}' \rightarrow \mathcal{F}$ is surjective if and only if the maps $\mathcal{F}'(x) \rightarrow \mathcal{F}(x)$ are surjective for all $x \in X$;
- (c) $\mathcal{F} = 0$ if and only if $\mathcal{F}(x) = 0$ for all $x \in X$.

Moreover, in the case of a scheme, it is enough to consider only closed points in (b) and (c).

Proof. Let $x \in X$ and let $M = \mathcal{F}_x$, which is a finitely generated module over the local ring $\mathcal{O}_{X,x}$, and then $\mathcal{F}(x) = M \otimes \kappa(x)$ is of finite dimension, showing (a). By Nakayama's lemma, a map of $\mathcal{O}_{X,x}$ -modules $N \rightarrow M$ is surjective if and only if $N \otimes \kappa(x) \rightarrow M \otimes \kappa(x)$ is surjective. Since a map of sheaves is surjective if and only if it is surjective on stalks, this shows (b), and (c) follows as a special case $\mathcal{F}' = 0$. As for the "moreover" part, we may assume that X is affine and that $\mathcal{F} = \tilde{M}$ for a finitely generated module M over $A = \mathcal{O}(X)$. The set $\text{supp}(M)$ of all primes $\mathfrak{p} \subseteq A$ such that $M_{\mathfrak{p}} \neq 0$ is a closed subset of $\text{Spec}(A)$ (as it is equal to $V(\text{Ann}(M))$ where $\text{Ann}(M) = \{a \in A : aM = 0\}$). This shows the stronger form of (c), and the stronger form of (a) follows similarly if we consider the cokernel of $\mathcal{F}' \rightarrow \mathcal{F}$ in place of \mathcal{F} . \square

11.3. Affine and finite morphisms

Recall from Lecture 9 that a morphism of schemes or algebraic sets $\phi: Y \rightarrow X$ is **affine** if for every affine $U \subseteq X$, the preimage $\phi^{-1}(U) \subseteq Y$ is affine. It is **finite** if moreover $\mathcal{O}(\phi^{-1}(U))$ is a finite $\mathcal{O}(U)$ -module for every affine $U \subseteq X$.

Proposition 11.3.1. (a) *Let X be a scheme. For every affine morphism $\phi: Y \rightarrow X$, the push-forward $\phi_*\mathcal{O}_Y$ is a quasi-coherent \mathcal{O}_X -algebra. The construction $(\phi: Y \rightarrow X) \mapsto \phi_*\mathcal{O}_Y$ defines an equivalence of categories*

$$\{\text{affine morphisms } Y \rightarrow X\}^{\text{op}} \xrightarrow{\sim} \{\text{quasi-coherent } \mathcal{O}_X\text{-algebras}\}.$$

(b) *If X is locally noetherian, the same construction gives an equivalence*

$$\{\text{finite morphisms } Y \rightarrow X\}^{\text{op}} \xrightarrow{\sim} \{\text{coherent } \mathcal{O}_X\text{-algebras}\}.$$

(c) *A morphism $\phi: Y \rightarrow X$ is affine (resp. finite) if and only if there exists an affine open cover $X = \bigcup U_\alpha$ such that the preimages $\phi^{-1}(U_\alpha)$ are affine (resp. and $c\mathcal{O}(\phi^{-1}(U_\alpha))$ is finite over $\mathcal{O}(U_\alpha)$) for every α .*

(d) *In particular, if X is affine, then $Y \rightarrow X$ is affine (resp. finite) if and only if Y is affine (resp. and $\mathcal{O}(Y)$ is finite over $\mathcal{O}(X)$).*

The inverse functor is called the **relative spectrum** $\mathcal{A} \mapsto \mathbf{Spec}_X(\mathcal{A})$.

Remark 11.3.2. Let X be an algebraic set. Then similarly we have equivalences as above,

$$\{\text{affine morphisms } Y \rightarrow X\}^{\text{op}} \xrightarrow{\sim} \{\text{reduced quasi-coherent } \mathcal{O}_X\text{-algebras of finite type}\}.$$

and

$$\{\text{finite morphisms } Y \rightarrow X\}^{\text{op}} \xrightarrow{\sim} \{\text{reduced coherent } \mathcal{O}_X\text{-algebras}\}.$$

Example 11.3.3 (Normalization). Let X be an integral scheme or an irreducible algebraic set, let $K = k(X)$ be its function field, and let L/K be a finite extension of fields. Consider the \mathcal{O}_X -algebra \mathcal{A} defined on non-empty affine opens $U \subseteq X$ by

$$\mathcal{A}(U) = \text{integral closure of } \mathcal{O}(U) \text{ in } L.$$

Then \mathcal{A} is a quasi-coherent algebra, and the corresponding affine morphism $Y \rightarrow X$ is called the **normalization of X in L** . By “finiteness of integral closure,” if X is an irreducible algebraic set, a scheme of finite type over a field, or a locally noetherian scheme and L/K is separable, then the morphism $Y \rightarrow X$ is finite. We shall primarily use this construction in two cases: $L = K$ (in which case we call Y the normalization of X), or when X is a smooth curve.

Example 11.3.4. Let $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the projection map. The preimage of the standard affine open $U_i = D(x_i) \simeq \mathbb{A}^n$ is the open subset $D(x_i) \subseteq \mathbb{A}^{n+1}$. Thus π is affine. The corresponding coherent $\mathcal{O}_{\mathbb{P}^n}$ -algebra is the direct sum $\bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$ (see the next lecture).

12. Lecture 12 (Mar 10): Ideal sheaves and locally free sheaves

Recommended reading: Kempf §5, Hartshorne II.5

12.1. Closed immersions and sheaves of ideals

A **quasi-coherent ideal** is a quasi-coherent \mathcal{O}_X -submodule \mathcal{J} of \mathcal{O}_X . The quotient $\mathcal{O}_X/\mathcal{J}$ is then a quasi-coherent \mathcal{O}_X -algebra. If X is a locally noetherian scheme or an algebraic set, then every quasi-coherent ideal of \mathcal{O}_X is coherent.

Lemma 12.1.1. *Let $i: Y \rightarrow X$ be a closed immersion. Then the kernel \mathcal{J}_Y of the surjection $\mathcal{O}_X \rightarrow i_*\mathcal{O}_Y$ is a quasi-coherent ideal. Moreover, i is a finite morphism. Conversely, for every quasi-coherent ideal \mathcal{J} there exists a unique closed immersion $i: Y \rightarrow X$ for which $i_*\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$.*

Proof. The sheaf $i_*\mathcal{O}_Y$ is quasi-coherent by Lemma 11.1.3, and hence \mathcal{J} is quasi-coherent, being the kernel of a map between quasi-coherent sheaves. Knowing \mathcal{J} , we can reconstruct Y as the support of $\mathcal{O}_X/\mathcal{J}$ (a closed subset of X) together with the restriction \mathcal{O}_Y of the sheaf $\mathcal{O}_X/\mathcal{J}$. On an affine open $U = \text{Spec}(A) \subseteq X$, we have $i^{-1}(U) \simeq \text{Spec}(A/I)$ where $I = \mathcal{J}(U)$. In particular, i is an affine (and hence finite) morphism. \square

Corollary 12.1.2. *If $X = \text{Spec}(A)$, then every closed subscheme of X is of the form $\text{Spec}(A/I)$ for a unique ideal $I \subseteq A$.*

12.2. Locally free sheaves (a.k.a. vector bundles)

A locally free \mathcal{O}_X -module is an algebraic analog of a vector bundle over X .

Definition 12.2.1. Let X be a ringed space. An \mathcal{O} -module \mathcal{E} is **locally free**⁶ if it is locally isomorphic to \mathcal{O}_X^n for some integer $n \geq 0$. More precisely, if there exists an open cover $X = \bigcup U_\alpha$, integers $n_\alpha \geq 0$, and isomorphisms $\mathcal{E}|_{U_\alpha} \simeq \mathcal{O}_{U_\alpha}^{n_\alpha}$. If all n_α are equal to a single integer n , we say that \mathcal{E} has **rank** n .

Lemma 12.2.2. *Let X be a locally noetherian scheme or an algebraic set and let \mathcal{E} be a coherent sheaf on X . The following conditions are equivalent:*

- (a) \mathcal{E} is locally free;
- (b) for every $x \in X$, the stalk \mathcal{E}_x is a free $\mathcal{O}_{X,x}$ -module;
- (c) for every affine open $U \subseteq X$, $\mathcal{E}(U)$ is a projective $\mathcal{O}(U)$ -module.

If X is reduced (or an algebraic set), these are further equivalent to

- (d) The function

$$\rho(x) = \dim_{\kappa(x)} \mathcal{E}(x)$$

is locally constant.

Example 12.2.3. The most natural example of a locally free sheaf is the sheaf of algebraic differential forms Ω_X^1 on a smooth variety X (see Lecture 17).

Remark 12.2.4. Tensor operations from (multi)linear algebra (duals, tensor product, symmetric and exterior powers) all apply to locally free sheaves without change.

Remark 12.2.5. The algebra of polynomial functions on a vector space V is the symmetric algebra $\text{Sym}(V^\vee)$. We can therefore turn V into a scheme $\text{Spec}(\text{Sym}(V^\vee))$. Using this recipe, we can turn a locally free sheaf \mathcal{E} into an honest vector bundle $E \rightarrow X$, a map whose fiber at $x \in X$ is naturally isomorphic to the $\kappa(x)$ -vector space $E(x)$, as follows:

$$E = \text{Spec}_X(\text{Sym}(\mathcal{E}^\vee)).$$

⁶This terminology may be non-standard, as we require our locally free sheaves to have locally finite rank.

12.3. Invertible sheaves (a.k.a. line bundles)

Invertible sheaves are locally free sheaves of rank one, and therefore correspond (and are often called) line bundles.

Definition 12.3.1. An **invertible sheaf** is a locally free sheaf of rank one, i.e. an \mathcal{O}_X -module \mathcal{L} locally isomorphic to \mathcal{O}_X .

The terminology “invertible” comes from the fact that invertible sheaves are invertible elements of the monoid of (isomorphism classes of) \mathcal{O}_X -modules under the tensor product. Namely, if \mathcal{L} is an invertible sheaf, then so is $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, and the “evaluation map” $\mathcal{L} \otimes \mathcal{L}^\vee \rightarrow \mathcal{O}_X$ is an isomorphism.

If \mathcal{L} and \mathcal{M} are invertible sheaves, then so is their tensor product $\mathcal{L} \otimes \mathcal{M}$. Thus, the isomorphism classes of invertible sheaves on X form an abelian group (with neutral element \mathcal{O}_X and tensor product as the group operation) called the **Picard group** of X and denoted by $\text{Pic}(X)$. Thinking in terms of this group, it is reasonable to introduce the following notation: for an invertible sheaf \mathcal{L} and an integer $n \in \mathbb{Z}$, we set $\mathcal{L}^n = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ (n factors) if $n \geq 1$, $\mathcal{L}^0 = \mathcal{O}_X$, and $\mathcal{L}^n = (\mathcal{L}^\vee)^{-n}$ if $n < 0$. We shall sometimes use the notation $\mathcal{L}^{\otimes n}$ for \mathcal{L}^n , to avoid confusing it with the direct sum $\mathcal{L}^{\oplus n} = \mathcal{L} \oplus \cdots \oplus \mathcal{L}$.

Remark 12.3.2 (Invertible sheaves via cocycles). Invertible sheaves are conveniently described in terms of Čech cocycles valued in the sheaf \mathcal{O}_X^\times of nowhere vanishing functions. Namely, if \mathcal{L} is an invertible sheaf, we can find an open cover $X = \bigcup U_\alpha$ and isomorphisms $\phi_\alpha: \mathcal{O}_{U_\alpha} \xrightarrow{\sim} \mathcal{L}|_{U_\alpha}$. Then on $U_\alpha \cap U_\beta$, the composition $\phi_\beta^{-1} \phi_\alpha$ gives an \mathcal{O} -module isomorphism $\mathcal{O}_{U_\alpha \cap U_\beta} \rightarrow \mathcal{O}_{U_\alpha \cap U_\beta}$, which is the multiplication by a nonvanishing function $g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$. The tuple of functions $\{g_{\alpha\beta}\}$ satisfies the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. Such a tuple is called a Čech cocycle valued in the sheaf \mathcal{O}^\times . Conversely, given an open cover $X = \bigcup U_\alpha$ and a Čech cocycle $\{g_{\alpha\beta}\}$, we can construct an invertible sheaf \mathcal{L} by gluing together the sheaves \mathcal{O}_{U_α} along the isomorphism $g_{\alpha\beta}$ on the overlaps $U_\alpha \cap U_\beta$. If \mathcal{L} and \mathcal{M} are two invertible sheaves, given by cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ subordinate to the same open cover $X = \bigcup U_\alpha$, then the invertible sheaves \mathcal{L}^n and $\mathcal{L} \otimes \mathcal{M}$ are given by the cocycles

$$\{g_{\alpha\beta}^n\} \quad \text{and} \quad \{g_{\alpha\beta}h_{\alpha\beta}\},$$

respectively. (Once we learn about cohomology, a more detailed analysis of this will show that $\text{Pic}(X)$ is naturally isomorphic to the cohomology group $H^1(X, \mathcal{O}_X^\times)$.)

Example 12.3.3 (The sheaves $\mathcal{O}(d)$ on \mathbb{P}^n). Let $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the projection map and let d be an integer. For an open subset $V \subseteq \mathbb{A}^{n+1}$ which is invariant under the action of k^\times by scaling, we say that a function $f \in \mathcal{O}(V)$ is homogeneous of degree d if $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ holds for all $\lambda \in k^\times$ and all $(x_0, \dots, x_n) \in V$. For an open subset $U \subseteq \mathbb{P}^n$, we set

$$\mathcal{O}(d)(U) = \{f \in \mathcal{O}(\pi^{-1}(U)) : f \text{ is homogeneous of degree } d\}.$$

This is a sheaf, by construction a subsheaf of $\pi_*\mathcal{O}$.

Let us compute $\mathcal{O}(d)(D_{\mathbb{P}}(f))$ for a distinguished affine open $D_{\mathbb{P}}(f) \subseteq \mathbb{P}^n$, with $f \in k[x_0, \dots, x_n]$ homogeneous of degree $e > 0$. We have $\pi^{-1}(D_{\mathbb{P}}(f)) = D(f)$, the distinguished affine open in \mathbb{A}^{n+1} , and hence

$$\mathcal{O}(\pi^{-1}(D_{\mathbb{P}}(f))) = k[x_0, \dots, x_n][f^{-1}].$$

This is naturally a graded ring, with homogeneous elements of degree d of the form h/f^m where $h \in k[x_0, \dots, x_n]$ is homogeneous of degree $d + me$.

In the special case $f = x_0$, we have

$$\mathcal{O}(d)(D(x_0)) = k[x_0, \dots, x_n][x_0^{-1}]_d = x_0^d k[x_1/x_0, \dots, x_n/x_0],$$

a free $\mathcal{O}(D(x_0))$ -module of rank one.

We deduce from these computations that $\pi_*\mathcal{O} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)$ and that each $\mathcal{O}(d)$ is an invertible sheaf. We shall study them in greater detail in Lecture 13.

13. Lecture 13 (Mar 12): Invertible sheaves on affine and projective spaces

Recommended reading: Vakil 15.1

13.1. Preview: the Riemann hypothesis for curves over finite fields

We start with a discussion (unrelated to where we are in the course) of point counting problems over fields.

Basic facts about finite fields:

- for every prime p and every $e \geq 1$ there exists a (unique up to isomorphism) field \mathbb{F}_q with $q = p^e$ elements;
- finite extensions of \mathbb{F}_q are of the form \mathbb{F}_{q^n} for some $n \geq 1$ (it is the splitting field of the separable polynomial $x^{q^n} - x$).

Let $k = \mathbb{F}_q$ be a finite field with $q = p^e$ elements, let \bar{k} be a fixed algebraic closure of k , and let X be a projective variety over k . Since we have not discussed varieties over non-algebraically closed field, let us simply say that we have a projective variety X over \bar{k} which is the zero set in \mathbb{P}^n of a homogeneous ideal $I \subseteq \bar{k}[x_0, \dots, x_n]$ which is generated by elements of $k[x_0, \dots, x_n]$ (in simple words, X is defined in $\mathbb{P}^n_{\bar{k}}$ by a system of homogeneous equations with coefficients in the finite field k). By $X(k)$ we denote the set of points of X with coordinates in k (called the set of k -rational points of X). It is a finite set, and we denote its cardinality by $\#X(k)$.

We will be interested in the case when X is a smooth curve. Its **genus** $g = g(X)$ is a non-negative integer which will be defined later (as the dimension of the space $\Gamma(X, \Omega_X^1)$ of differentials on X), for now it is enough to know that if X is a plane curve, cut out of \mathbb{P}^2 by a homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d , then

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

Thus, the genus of a line, conic, cubic, quartic is 0, 0, 1, 3. (Since there exist curves of every genus $g \geq 0$, as we shall see later, it follows from the asymptotics of the above formula that most curves do not embed into \mathbb{P}^2 .) We are interested in the size of $X(k)$. As a first approximation, it makes sense to compare it to the size of $\mathbb{P}^1(k)$, which equals $q + 1$.

Theorem 13.1.1 (“Riemann hypothesis”, Hasse, Schmidt, Weil). *Suppose that X is a smooth and projective connected curve defined over \mathbb{F}_q , and let g be its genus. We have*

$$|\#X(k) - q - 1| \leq 2g\sqrt{q}.$$

It is called the Riemann hypothesis for the following reason. For any variety X over $k = \mathbb{F}_q$ one can organize the numbers $\#X(\mathbb{F}_{q^n})$ for varying $n \geq 1$ into the **Hasse–Weil zeta function**

$$\zeta_X(s) = \exp \left(\sum_{n \geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} q^{-ns} \right).$$

One can then show (by a theorem of Dwork, later reproved by Grothendieck) that $\zeta_X(s)$ is always a rational function of q^{-s} . Theorem 13.1.1 can then be seen to be equivalent to the statement that for X a smooth projective curve all zeros of $\zeta_X(s)$ lie on the line $\operatorname{Re}(s) = \frac{1}{2}$. Deligne has later generalized Theorem 13.1.1 to higher dimensional varieties.

Exercise 13.1.2. Compute $\zeta_{\mathbb{P}^n}(s)$.

We want to develop enough theory this term to prove Theorem 13.1.1. Let us give a quick preview how this will go. Notice first that $k = \mathbb{A}^1(k)$ is the solution set of the equation $x^q = x$ in \bar{k} , or equivalently the fixed-point set of the map $F: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ defined by $F(x) = x^q$. Note that $x^q - x = \prod_{\lambda \in \mathbb{F}_q} (x - \lambda)$, so there are no multiple roots, i.e. the fixed points all have multiplicity one. We can generalize this to our curve $X \subseteq \mathbb{P}^N$ by noting that the map

$$F: \mathbb{P}^N \rightarrow \mathbb{P}^N, \quad F(x_0 : \cdots : x_N) = (x_0^q : \cdots : x_N^q)$$

maps X to X . Denoting the resulting map also by $F: X \rightarrow X$, we have

$$X(\mathbb{F}_q) = \text{Fix}(F), \quad X(\mathbb{F}_{q^n}) = \text{Fix}(F^n).$$

The cardinalities of these sets can be realized geometrically as intersection numbers on the surface $X \times X$: we have

$$\#\text{Fix}(F^n) = (\Delta_X \cdot \Gamma_{F^n}),$$

the intersection multiplicity of two curves: the diagonal $\Delta_X \subseteq X \times X$ and the graph Γ_{F^n} of F^n . Algebraic geometry allows you to control these numbers by means of intersection theory, which is based on properties of invertible sheaves on curves.

13.2. Invertible sheaves on \mathbb{A}^1 and \mathbb{P}^1

By a locally free sheaf we shall mean a locally free *coherent* sheaf, i.e. an \mathcal{O}_X -module which is locally isomorphic to \mathcal{O}_X^n for some integer $n \geq 0$. As before, we shall use the terms “locally free (coherent) sheaf” and “vector bundle” interchangeably. Similarly, an invertible sheaf will often be called a “line bundle.”

Lemma 13.2.1 (Vector bundles on \mathbb{A}^1). *Every locally free sheaf on \mathbb{A}^1 is free (isomorphic to $\mathcal{O}_{\mathbb{A}^1}^r$ for some $r \geq 0$). In particular, $\text{Pic}(\mathbb{A}^1) = 0$.*

Proof. Let \mathcal{E} be a locally free sheaf on \mathbb{A}^1 . Since $\mathbb{A}^1 = \text{Spec}(k[T])$ is affine, we have $\mathcal{E} \simeq \tilde{M}$ for some finitely generated $k[T]$ -module M . By the structure theorem for f.g. modules over PIDs we can write

$$M \simeq k[T]^{\oplus r} \oplus \bigoplus_{i=1}^m k[T]/(f_i), \quad f_1, \dots, f_r \in k[T], \quad \deg(f_i) > 0.$$

We claim that $m = 0$ (so that M , and hence also \mathcal{E} , is free). Otherwise let $x \in k$ be a zero of f_1 and let $\mathfrak{m} = (T - x)$ be the corresponding maximal ideal. Then $\mathcal{E}_x \simeq M_{\mathfrak{m}}$, which is not a free module over $\mathcal{O}_{\mathbb{A}^1, x} \simeq k[T]_{\mathfrak{m}}$ since it has torsion. \square

Construction. Let now $X = \mathbb{P}^1$ which we write as

$$X = U_0 \cup U_1, \quad U_0 = D(x_0) = \text{Spec}(k[y_1]), \quad U_1 = D(x_1) = \text{Spec}(k[y_0]).$$

We have

$$U_{01} := U_0 \cap U_1 = \text{Spec}(k[y_0, y_1]/(y_0 y_1 - 1)),$$

which is a Laurent polynomial ring in either y_0 or y_1 . In particular,

$$\mathcal{O}^\times(U_{01}) = k^\times \times y_0^{\mathbb{Z}} = k^\times \times y_1^{\mathbb{Z}}.$$

For $g_{01} = \lambda y_0^d \in \mathcal{O}^\times(U_{01})$ we define an invertible sheaf $\mathcal{L}(g_{01})$ on \mathbb{P}^1 by gluing together the trivial invertible sheaves \mathcal{O}_{U_i} on U_i , $i = 0, 1$ along the isomorphism

$$\mathcal{O}_{U_0}|_{U_{01}} = \mathcal{O}_{U_{01}} \xrightarrow{g_{01}} \mathcal{O}_{U_{01}} = \mathcal{O}_{U_1}|_{U_{01}}.$$

Lemma 13.2.2 (Line bundles on \mathbb{P}^1).

- (a) Every invertible sheaf on \mathbb{P}^1 is of the form $\mathcal{L}(g_{01})$ for some $g_{01} \in \mathcal{O}^\times(U_{01})$.
- (b) We have $\mathcal{L}(g_{01}) \otimes \mathcal{L}(g'_{01}) \simeq \mathcal{L}(g_{01}g'_{01})$.
- (c) We have $\mathcal{L}(g_{01}) \simeq \mathcal{O}_{\mathbb{P}^1}$ if and only if $g_{01} \in k^\times$.

Definition 13.2.3. For $d \in \mathbb{Z}$ we define $\mathcal{O}_{\mathbb{P}^1}(d) = \mathcal{L}(y_0^d)$.

Corollary 13.2.4. We have $\text{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$, generated by the class of $\mathcal{O}_{\mathbb{P}^1}(1)$.

Proof of Lemma 13.2.2. (a) Follows from Lemma 13.2.1. (b) Obvious once you unpack it and recall that the canonical isomorphism $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{O} \rightarrow \mathcal{O}$ is given by $f \otimes g \mapsto fg$. To prove (c), let us compute

$$\text{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{L}(g_{01})) = \Gamma(\mathbb{P}^1, \mathcal{L}(g_{01})).$$

A section $f \in \Gamma(\mathbb{P}^1, \mathcal{L}(g_{01}))$ corresponds to a pair of sections

$$f_i \in \Gamma(U_i, \mathcal{L}(g_{01})|_{U_i}) = \Gamma(U_i, \mathcal{O}_{U_i}) = k[y_{1-i}], \quad i = 0, 1$$

which agree on the overlap U_{01} , which translates into the condition

$$f_1 = g_{01}f_0.$$

Such a section f corresponds to an isomorphism $\mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{L}(g_{01})$ if and only if $f_i \in \mathcal{O}^\times(U_i)$ for $i = 0, 1$.

We can now show (\Leftarrow): if $g_{01} = \lambda \in k^\times$ is a nonzero constant, we can simply take $(f_0, f_1) = (1, \lambda)$. To show (\Rightarrow), we can now assume $g_{01} = y_0^d$ (this reduction is inessential for the calculation). We then have

$$\Gamma(\mathbb{P}^1, \mathcal{O}(d)) = \{f_0 \in k[y_1] : y_0^d f_0(y_0^{-1}) \in k[y_0]\}.$$

But for a polynomial $f \in k[y]$, we have that $y^d f(y^{-1})$ is also a polynomial if and only if $\deg(f) \leq d$. So

$$\Gamma(\mathbb{P}^1, \mathcal{O}(d)) = k[y_1]_{\leq d}$$

is the space of polynomials of degree $\leq d$, which can be identified more intrinsically/symmetrically with the space $k[x_0, x_1]_d$ of homogeneous polynomials of degree d . In particular,

$$\dim \Gamma(\mathbb{P}^1, \mathcal{O}(d)) = \dim k[x_0, x_1]_d = \max\{0, d + 1\}.$$

Since $\Gamma(\mathbb{P}^1, \mathcal{O})$ has dimension one, we conclude that $\mathcal{O}(d)$ is not isomorphic to \mathcal{O} for $d \neq 0$, showing (\Leftarrow). \square

Remark 13.2.5 (Vector bundles on \mathbb{P}^1). Given that Lemma 13.2.1 describes not only invertible sheaves but all locally free sheaves, one is led to ask what are the locally free sheaves on \mathbb{P}^1 . The answer is given by the following theorem of Birkhoff and independently Grothendieck: every locally free sheaf on \mathbb{P}^1 is isomorphic to the direct sum of $\mathcal{O}(d_i)$ for some integers $d_1, \dots, d_r \in \mathbb{Z}$. If we follow our strategy of proof of Lemma 13.2.2, we will see that this theorem is equivalent to the following elementary (but still non-obvious) statement: *given a matrix $G_{01} \in \text{GL}_r(k[y, y^{-1}])$, there exist integers d_1, \dots, d_r and two matrices $F_0 \in \text{GL}_r(k[y])$, $F_1 \in \text{GL}_r(k[y^{-1}])$ such that*

$$F_0 G_{01} F_1 = \text{diag}(y^{d_1}, \dots, y^{d_r}).$$

13.3. Invertible sheaves on \mathbb{A}^n and \mathbb{P}^n

We now redo everything we did in the previous subsection for \mathbb{A}^n and \mathbb{P}^n . This is a bit more tricky but the results are essentially the same.

Lemma 13.3.1 (Line bundles on \mathbb{A}^n). *We have $\text{Pic}(\mathbb{A}^n) = 0$.*

Proof (sketch). The proof relies on the following (not so difficult) fact from commutative algebra: *Every invertible (nonzero and locally principal) in a unique factorization domain is principal.* We can apply it since $A := \mathcal{O}(\mathbb{A}^n) = k[x_1, \dots, x_n]$ is a UFD. Let \mathcal{L} be an invertible sheaf on \mathbb{A}^n and write it as \tilde{L} for an A -module L . Let $x \in L$ be a nonzero element, corresponding to a map $A \rightarrow L$. We dualize it (apply $\text{Hom}(-, A)$) to get a map $L^\vee \rightarrow A$ which is injective (as this can be checked locally, whereupon $L = A$ and the map is multiplication by a nonzero $x \in A$). The image $I \subseteq A$ of this map is then a nonzero locally principal ideal (again, check locally), and hence principal. Thus $L^\vee \simeq I$ is free and hence so is L . \square

Remark 13.3.2 (Vector bundles on \mathbb{A}^n). Given that Lemma 13.2.1 describes not only invertible sheaves but all locally free sheaves, one is led to ask what are the locally free sheaves on \mathbb{A}^n . The answer is given by the following theorem of Quillen and Suslin (answering a long-standing conjecture of Serre): every locally free sheaf on \mathbb{A}^n is free. Equivalently, every finitely generated module over the polynomial ring is free.

Construction. Let now $X = \mathbb{P}^n$, which we write as

$$X = \bigcup_{i=0}^n U_i, \quad U_i = D(x_i) = \text{Spec}(k[y_{0/i}, \dots, y_{n/i}]/(y_{i/i} - 1))$$

using Vakil's handy notation $y_{j/i}$ for the regular function x_j/x_i on U_i . For a non-empty subset $I \subseteq \{0, \dots, n\}$ we write

$$U_I = \bigcap_{i \in I} U_i = \text{Spec} \left(\frac{k[y_{j/i} \text{ (for } j \in \{0, \dots, n\}, i \in I)]}{(y_{i/i} - 1 \text{ (for } i \in I), y_{j/i} y_{i'/i} - y_{j/i'} \text{ (for } j \in \{0, \dots, n\}, i, i' \in I))} \right).$$

This is isomorphic (after permuting the variables so that $I = \{0, \dots, \#I - 1\}$) to $\mathbb{G}_m^{\#I-1} \times \mathbb{A}^{n+1-\#I}$. We have

$$\mathcal{O}^\times(U_I) = k^\times \times \prod_{i < i'} y_{i/i'}^{\mathbb{Z}}.$$

In order to describe an invertible sheaf \mathcal{L} on \mathbb{P}^n , it suffices to describe its restrictions $\mathcal{L}_i = \mathcal{L}|_{U_i}$ and isomorphisms $\phi_{ij}: \mathcal{L}_i|_{U_{ij}} \rightarrow \mathcal{L}_j|_{U_{ij}}$ (with $\phi_{ii} = \text{id}$) which are "transitive" i.e. which satisfy the cocycle condition

$$\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \text{id} \quad \text{on } \mathcal{L}|_{U_{ijk}}$$

(this condition did not appear for \mathbb{P}^1 since we had only two open sets and no interesting triple overlaps). Now thanks to 13.3.1 we have $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$ (this isomorphism is canonical up to $\text{Aut}(\mathcal{O}_{U_i}) = \mathcal{O}^\times(U_i) = k^\times$). After making such an identification, the maps ϕ_{ij} are given by multiplication by some $g_{ij} \in \mathcal{O}^\times(U_{ij}) = k^\times \times y_{i/j}^{\mathbb{Z}}$, satisfying $g_{ii} = 1$ and $g_{ij} g_{jk} g_{ki} = 1$ on U_{ijk} . If we change each identification $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$ by an $f_i \in \mathcal{O}^\times(U_i)$, the g_{ij} are replaced with $g_{ij} \cdot (f_i/f_j)$.

Remark 13.3.3. The above considerations show that for every scheme or algebraic set X and an open cover $X = \bigcup U_i$ such that $\text{Pic}(U_i) = 0$, the Picard group $\text{Pic}(X)$ is the cohomology of the two-term sequence

$$\begin{array}{ccccc} & (g_{ij}) & \mapsto & (g_{ij} g_{jk} g_{ki}) & \\ & \cap & & \cap & \\ \prod_i \mathcal{O}^\times(U_i) & \longrightarrow & \prod_{ij} \mathcal{O}^\times(U_{ij}) & \longrightarrow & \prod_{ijk} \mathcal{O}^\times(U_{ijk}) \\ \downarrow & & \downarrow & & \\ (f_i) & \mapsto & (f_i/f_j) & & \end{array}$$

This sequence is part of the Čech complex for the sheaf \mathcal{O}^\times and the open cover $\{U_i\}$. Once we discuss cohomology, we shall relate this to the identification

$$\text{Pic}(X) \simeq H^1(X, \mathcal{O}^\times)$$

with the first cohomology group of \mathcal{O}^\times , valid with no assumptions on $\text{Pic}(U_i)$.

We now finish our construction.

Definition 13.3.4. Let $d \in \mathbb{Z}$. We define the invertible sheaf $\mathcal{O}(d)$ on \mathbb{P}^n using the functions $(g_{ij}) \in \mathcal{O}^\times(U_{ij})$ given by

$$g_{ij} = (x_i/x_j)^d = y_{i/j}^d = y_{j/i}^{-d}.$$

It is clear that the cocycle condition is satisfied.

Lemma 13.3.5. Let $n \geq 1$.

- (a) Every invertible sheaf on \mathbb{P}^n is of the form $\mathcal{O}_{\mathbb{P}^n}(d)$ for a unique $d \in \mathbb{Z}$.
- (b) We have $\mathcal{O}(d) \otimes \mathcal{O}(d') \simeq \mathcal{O}(d + d')$.
- (c) We have

$$\Gamma(\mathbb{P}^n, \mathcal{O}(d)) \simeq k[x_0, \dots, x_n]_d$$

which is a k -vector space of dimension $\binom{n+d}{d}$.

Proof. (b) is clear, and (c) is a direct calculation which we omit. From (c) it follows that $\dim \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ equals 1 only for $d \neq 0$, showing uniqueness in (a). The remainder of (a) will be shown later, once we discuss the relationship between invertible sheaves and divisors. \square

Remark 13.3.6 (Vector bundles on \mathbb{P}^n). Given that Remark 13.2.5 describes all locally free sheaves on \mathbb{P}^1 and Lemma 13.3.5 says that all invertible sheaves are of the form $\mathcal{O}(d)$, one may wonder what one can say about higher rank locally free sheaves on \mathbb{P}^n for $n > 1$. The answer, in brief, is that vector bundles on \mathbb{P}^n of rank r are **hard** to describe as soon as $n > 1$ or $r > 1$. In particular, not every locally free sheaf on \mathbb{P}^n is **split**, i.e. of the form $\mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_r)$ for some integers d_1, \dots, d_r . For example, the cotangent bundle (sheaf of Kähler differentials) $\Omega_{\mathbb{P}^n}^1$ is of rank n and is not split if $n > 1$ (this is very easy to prove using coherent cohomology, a topic we shall treat next). In fact, already rank two vector bundles on \mathbb{P}^2 can vary in families, by which we mean: there exists a positive-dimensional variety S and a rank two locally free sheaf \mathcal{E} on $\mathbb{P}^2 \times S$ such that for every two distinct points $s_0, s_1 \in S$ the restrictions of \mathcal{E} to $\{s_i\} \times \mathbb{P}^2$ ($i = 0, 1$) are not isomorphic as locally free sheaves on \mathbb{P}^2 . In contrast, an important and widely open conjecture of Hartshorne states that rank two vector bundles on \mathbb{P}^n are split as long as $n \geq 5$. It is rather embarrassing that we lack an answer to such a simple looking question.

14. Lecture 14 (Mar 17): Divisors

Recommended reading: Hartshorne II.6, Kempf 5.5.

14.1. Overview

Let X be an algebraic set (over our algebraically closed field k). We shall work under the following simplifying assumptions:

X is a smooth variety (separated and irreducible).

(See Remark 14.3.7 below for a brief discussion what needs to be changed if we relax these conditions.) The assumption implies that the local rings $\mathcal{O}_{X,x}$ (or more generally the stalks $\mathcal{O}_{X,Z}$ for irreducible $Z \subseteq X$) are regular. Since regular rings are UFDs (as already mentioned in Lecture 6, §6.2), they are integrally closed, and hence for every non-empty affine $U \subseteq X$, the algebra $\mathcal{O}(U)$ is an integrally closed domain. For a big portion of the remainder of the course we shall be interested in the special case X being a smooth and projective curve, in which case the local rings are $\mathcal{O}_{X,x}$ are discrete valuation rings, and the algebras $\mathcal{O}(U)$ for $U \subseteq X$ non-empty affine are Dedekind domains.

A **prime divisor** on X is a codimension one irreducible closed subset $D \subseteq X$ (if X is a curve, this means D is simply a point of X). The local ring (see Lecture 5, Definition 5.5.1) $\mathcal{O}_{X,D}$ is then a discrete valuation ring (since it is one-dimensional and normal) with fraction field $k(X)$. We denote by

$$v_D : k(X) \longrightarrow \mathbb{Z} \cup \{+\infty\}$$

the associated valuation (i.e. $v_D(f)$ is the “order of vanishing” — or pole, if negative — of f along D). A **divisor** on X is a formal combination

$$D = \sum_{i=1}^r a_i D_i, \quad a_i \in \mathbb{Z}$$

of prime divisors D_i . To a rational function $f \in k(X)$ one can associate a divisor

$$\operatorname{div}(f) = \sum_D v_D(f) \cdot D \tag{14.1.1}$$

the sum taken over all prime divisors $D \subseteq X$ (Proposition 14.2.2(a) below ensures that this sum is finite). Such divisors are called **principal**. The main goal of the lecture is to describe the Picard group of isomorphism classes of invertible sheaves $\operatorname{Pic}(X)$ in terms of divisors.

Theorem 14.1.1. *There is a natural exact sequence*

$$1 \longrightarrow \mathcal{O}(X)^\times \longrightarrow k(X)^\times \xrightarrow{\operatorname{div}} \operatorname{Div}(X) \xrightarrow{\pi} \operatorname{Pic}(X) \longrightarrow 1. \tag{14.1.2}$$

Proof outline. The proof will occupy the next two subsections. Injectivity on the left is obvious, the map div is well-defined thanks to Proposition 14.2.2(a), and exactness at $k(X)^\times$ is proved in Proposition 14.2.2(c). The map π is defined in §14.3, and exactness at $\operatorname{Div}(X)$ is proved in Proposition 14.3.5. Finally, surjectivity of π is shown in Proposition 14.3.6. \square

In simple terms, Theorem 14.1.1 states that

$$\text{line bundles} = \frac{\text{divisors}}{\text{principal divisors}}$$

Two divisors D, D' are **linearly equivalent** (denoted $D \sim D'$) if $D - D'$ is principal. Thus line bundles are linear equivalence classes of divisors.

14.2. Divisors associated to rational functions

We introduce some basic terminology:

Definition 14.2.1. Let $D = \sum a_i D_i \in \text{Div}(X)$ be a divisor on X .

- (a) We say that D is **effective** if $a_i \geq 0$ for all i .
- (b) If $E \in \text{Div}(X)$ is another divisor, we write $D \geq E$ if $D - E$ is effective.
- (c) For an open $U \subseteq X$, we write $D|_U$ for the sum $\sum a_i (D_i \cap U)$ where we omit the indices i for which $D_i \cap U = \emptyset$.

Proposition 14.2.2. Let $f \in k(X)^\times$ be a nonzero rational function.

- (a) There are only finitely many prime divisors $D \subseteq X$ such that $v_D(f) \neq 0$.
- (b) The function f is regular (meaning that $f \in \mathcal{O}(X)$) if and only if $v_D(f) \geq 0$ for all prime divisors D .
- (c) The function f is invertible on X if and only if $v_D(f) = 0$ for all prime divisors D .

The first assertion means that the divisor $\text{div}(f)$ in (14.1.1) is well defined, and since $v_D(fg) = v_D(f) + v_D(g)$ we have a homomorphism

$$\text{div} : k(X)^\times \longrightarrow \text{Div}(X), \quad \text{div}(f) = \sum_{D \subseteq X \text{ prime}} v_D(f) \cdot D.$$

Part (b) means that regular functions are those which map to effective divisors, and part (c) that invertible functions comprise the kernel. In particular, we have an exact sequence (the left part of (14.1.2))

$$1 \longrightarrow \mathcal{O}(X)^\times \longrightarrow k(X)^\times \xrightarrow{\text{div}} \text{Div}(X).$$

Proof of Proposition 14.2.2. To prove (a) we may assume that X is affine and that $f \in A = \mathcal{O}(X)$ is a regular function (why?). In this case prime divisors $D \subseteq X$ correspond to height one prime ideals $\mathfrak{p} \subseteq A$. We have $v_D(f) \geq 0$ for all $f \in A$, and $v_D(f) > 0$ if and only if the image of f in $\mathcal{O}_{X,D} = A_{\mathfrak{p}}$ belongs to the maximal ideal $\mathfrak{m}_D = \mathfrak{p} \cdot A_{\mathfrak{p}}$, which happens precisely when $f \in \mathfrak{p}$. Since $\dim(A/\mathfrak{p}) = \dim(A) - 1$, the height one primes $\mathfrak{p} \subseteq A$ correspond to minimal primes of A/\mathfrak{p} , of which there are finitely many since A/\mathfrak{p} is noetherian.

To prove (b) we again reduce to the case X affine and let $A = \mathcal{O}(X)$. Using the above reasoning, the first claim is equivalent to the commutative algebra statement: *A is equal to the intersection, inside its field of fractions, of its localizations $A_{\mathfrak{p}}$ at all height one prime ideals.* The proof of this result, given in the appendix to this lecture (see Proposition 14.4.3), relies on A being integrally closed.

Part (c) follows from (b) since f belongs to $\mathcal{O}^\times(X)$ if and only if both f and f^{-1} are regular. \square

14.3. Invertible sheaves associated to divisors

Our next goal is to construct the map $\pi : \text{Div}(X) \rightarrow \text{Pic}(X)$ in (14.1.2), i.e. to associate to every divisor D on X an invertible sheaf, which we denote by $\mathcal{O}_X(D)$ (and which Hartshorne denoted by $\mathcal{L}(D)$). Intuitively, sections of $\mathcal{O}_X(D)$ corresponding to D are rational functions whose poles are ‘no worse than D .’ The following definition makes this precise.

Definition 14.3.1. Let D be a divisor on X . We define $\mathcal{O}_X(D)$ to be the following presheaf on X :

$$\mathcal{O}_X(D)(U) = \{0\} \cup \{f \in k(X)^\times : (\text{div}(f) + D)|_U \geq 0\} \subseteq k(X)$$

(if $U \neq \emptyset$). By definition, $\mathcal{O}_X(D)$ is a sub-presheaf of the constant sheaf \mathcal{K}_X with value $k(X)$.

Examples 14.3.2. (a) Let $D \subseteq X$ be a prime divisor and let $\mathcal{J}_D \subseteq \mathcal{O}_X$ be the corresponding coherent ideal. Then $\mathcal{O}_X(-D) = \mathcal{J}_D$. Consequently, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

The same holds for every effective divisor $D = \sum a_i D_i$, which we may identify with the closed subscheme $V(\mathcal{J}_D)$ for $\mathcal{J}_D = \prod \mathcal{J}_{D_i}^{a_i}$ (note that if $a_i > 1$ for some i then this is non-reduced, so if we do not work with schemes we cannot really consider D as a geometric subobject of X). Note that we have $D \geq 0$ if and only if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$, in which case we have a canonical section $1 \in \Gamma(X, \mathcal{O}_X(D))$.

A bit more generally, if D and E are divisors with $D \geq 0$, we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E - D) \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(E)|_D \longrightarrow 0.$$

Note that we cannot in general write $\mathcal{O}_X(E)|_D$ as $\mathcal{O}_D(E \cap D)$ since D and E may have a common component.

- (b) Let $X = \mathbb{P}^n$ for $n > 0$ and let $D \subseteq X$ be a prime divisor. Then $D = V(f)$ for an irreducible homogeneous $f \in k[x_0, \dots, x_n]$ of degree d . In this case we have $\mathcal{O}_X(D) \simeq \mathcal{O}(d)$. Combined with Theorem 14.1.1 (still to be proved), this completes our proof from Lecture 12 that $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$, generated by $\mathcal{O}(1)$.

Definition 14.3.1 introduced (implicitly) the sheaf of rational functions \mathcal{K}_X . Let us note that it is quasi-coherent: indeed, if X is affine, we have $\mathcal{K}_X = \widetilde{k(X)}$ (check this!). Since we are discussing invertible subsheaves of \mathcal{K}_X , let us carefully distinguish between *equality* ($=$) and *isomorphism* (\simeq).

Lemma 14.3.3. *Every divisor D on X is locally principal. That is, locally on X there exists a rational function $f \in k(X)^\times$ such that $\text{div}(f) = D$.*

Proof. Since divisors are combinations of prime divisors, and combinations of locally principal divisors are locally principal, it suffices to show that every prime divisor is locally principal. Let thus D be a prime divisor on X and let $x \in X$; we seek to show that D is principal in a neighborhood of x . If $x \notin D$ there is nothing to show, so let us assume $x \in D$. Let $A = \mathcal{O}_{X,x}$ which is regular (as X is smooth) and hence a UFD. As we have remarked already in Lecture 13, this implies that every height one prime ideal $\mathfrak{p} \subseteq \mathcal{O}_{X,x}$ is principal. Thus $\mathcal{J}_D \cdot \mathcal{O}_{X,x}$ is principal, generated by the image of a regular function $f \in \mathcal{O}(U)$ defined on an open affine neighborhood U of x . Shrinking U , we may ensure that $\mathcal{J}_D \cdot \mathcal{O}(U) = (f)$, so that $D|_U = \text{div}(f)|_U$ is principal. \square

Remark 14.3.4 (Cohomological interpretation). By the above lemma, a divisor can be described in terms of an open cover $X = \bigcup U_\alpha$ and nonzero rational functions f_α such that $g_{\alpha\beta} = f_\alpha/f_\beta \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$. If you think carefully what happens when we refine the cover, you will deduce from this an isomorphism

$$\text{Div}(X) \simeq \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times).$$

The functions $g_{\alpha\beta}$ form an \mathcal{O}^\times -valued cocycle which defines the line bundle $\mathcal{O}_X(D)$. Once we learn about cohomology, we will be able to recognize the sequence (14.1.2) as part of the cohomology exact sequence associated to

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{K}_X^\times \longrightarrow \mathcal{K}_X^\times / \mathcal{O}_X^\times \longrightarrow 1$$

which reads

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}_X^\times(X) & \longrightarrow & k(X)^\times & \longrightarrow & \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{K}_X^\times) \longrightarrow \dots \\ & & & & & & \parallel & \parallel & \parallel \\ & & & & & & \text{Div}(X) & \text{Pic}(X) & 1. \end{array}$$

Here the last group $H^1(X, \mathcal{K}_X^\times)$ is trivial since the sheaf \mathcal{K}_X^\times is flabby (being a constant sheaf on an irreducible space).

Proposition 14.3.5. *Let D be a divisor on X .*

(a) *The presheaf $\mathcal{O}_X(D)$ is an invertible subsheaf of the quasi-coherent sheaf \mathcal{K}_X .*

(b) *If $E \in \text{Div}(X)$ is another divisor, we have*

$$\mathcal{O}_X(D) \otimes \mathcal{O}_X(E) = \mathcal{O}_X(D) \cdot \mathcal{O}_X(E) = \mathcal{O}_X(D+E) \subseteq \mathcal{K}_X.$$

(c) *If $D = \text{div}(f)$ for a rational function $f \in k(X)^\times$, then $\mathcal{O}_X(D) = f^{-1} \cdot \mathcal{O}_X$. In particular, $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ in this case.*

(d) *Conversely, if $\mathcal{O}_X(D) \simeq \mathcal{O}_X$, then D is principal.*

(e) *Every invertible subsheaf of \mathcal{K}_X is of the form $\mathcal{O}_X(D)$ for a unique divisor D .*

Proof. Let us first show (c). By Proposition 14.2.2(b) we have

$$\mathcal{O}_X(\text{div}(f))(U) = \{g : \text{div}(fg) \geq 0\} = \{g : fg \in \mathcal{O}(U)\} = f^{-1}\mathcal{O}(U).$$

We show (a). This is a local property, so we may assume thanks to Lemma 14.3.3 that $D = \text{div}(f)$ is principal. In this case the assertion follows from (c). Part (b) is easy and left as an exercise.

To show (d), suppose we have an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X(D)$ and let $f \in \mathcal{O}_X(D)(X)$ be the image of 1 under this isomorphism, which is a nowhere-vanishing global section of $\mathcal{O}_X(D)$. We claim that $D = -\text{div}(f)$. This can easily be checked directly by looking at the local rings $\mathcal{O}_{X,P}$ for all prime divisors $P \subseteq X$.

The proof of (e) follows the same strategy: since both invertible subsheaves of \mathcal{K}_X and divisors can be defined locally (i.e. both form sheaves of sets on X), we can work locally, thus reducing the problem to the case of trivial (isomorphic to \mathcal{O}_X) invertible subsheaves of \mathcal{K}_X . But these are of the form $f \cdot \mathcal{O}_X$ for some rational function $f \in k(X)^\times$, and hence equal to $\mathcal{O}_X(D)$ where $D = \text{div}(f)$. Moreover, $f \cdot \mathcal{O}_X = \mathcal{O}_X$ precisely when f is invertible, i.e. when $D = 0$, which shows uniqueness. \square

Looking back at our target Theorem 14.1.1, defining $\pi : \text{Div}(X) \rightarrow \text{Pic}(X)$ as $D \mapsto \mathcal{O}_X(D)$, Proposition 14.3.5 gives the exactness of the following portion of the sequence

$$k(X)^\times \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\pi} \text{Pic}(X).$$

Therefore what is left to show is that the map π is surjective.

Proposition 14.3.6. *Let \mathcal{L} be an invertible sheaf on X . Then \mathcal{L} embeds into \mathcal{K}_X . Consequently, there exists a divisor D on X such that $\mathcal{L} \simeq \mathcal{O}_X(D)$.*

Proof. For every invertible sheaf \mathcal{L} on X we have

$$\text{Hom}(\mathcal{L}, \mathcal{K}_X) = \text{Hom}(\mathcal{O}_X, \mathcal{L}^{-1} \otimes \mathcal{K}_X) = \Gamma(X, \mathcal{L}^{-1} \otimes \mathcal{K}_X) = \varinjlim_{U \neq \emptyset} \mathcal{L}^{-1}(U).$$

An element of $\Gamma(X, \mathcal{L} \otimes \mathcal{K}_X)$ is called a **rational section** of \mathcal{L} . Note that this is a one-dimensional vector space over $k(X)$. Via the above identification, nonzero rational sections of \mathcal{L}^{-1} correspond to injective maps $\mathcal{L} \hookrightarrow \mathcal{K}_X$. \square

Remark 14.3.7 (Weil divisors vs Cartier divisors). Parts of what we have proved works under less restrictive assumptions. Here is a brief overview (for a more complete picture, see Hartshorne II.6).

- All results from this lecture work for any integral normal noetherian scheme which is locally factorial, meaning that the local rings $\mathcal{O}_{X,x}$ are UFDs (for example, X regular).

- On a general noetherian integral scheme X one has to distinguish between **Weil divisors** $\text{WDiv}(X)$, which are defined as here as integer combinations of codimension one closed irreducible subsets, and **Cartier divisors** $\text{CDiv}(X)$, which are the locally principal ones.
- The map $\text{div}: k(X)^\times \rightarrow \text{CDiv}(X)$ is well defined as long as the local rings $\mathcal{O}_{X,\xi}$ at codimension one points $\xi \in X$ are discrete valuation rings. This will hold for example if X is normal. In this case we define the **divisor class group** $\text{Cl}(X)$ as the quotient

$$\text{Cl}(X) = \text{WDiv}(X)/\text{im}(\text{div}: k(X)^\times \rightarrow \text{CDiv}(X))$$

of Weil divisors by principal ones. Under the same assumptions,

$$\text{Pic}(X) = \text{CDiv}(X)/\text{im}(\text{div}: k(X)^\times \rightarrow \text{CDiv}(X))$$

which is therefore a subgroup of $\text{Cl}(X)$.

Exercise 14.3.8. Suppose that X is projective and let \mathcal{L} be an invertible sheaf on X . Prove that if both \mathcal{L} and \mathcal{L}^{-1} have nonzero global sections, then \mathcal{L} is trivial.

14.4. Appendix: commutative algebra background

Proposition 14.4.1 (Stacks Project 0AFT). *A noetherian domain is a UFD if and only every height prime is principal.*

Proof. We only prove the “only if” part, which is what we need for our treatment of divisors. Let A be a noetherian UFD and let $\mathfrak{p} \subseteq A$ be a height one prime ideal. Let $x \in \mathfrak{p}$ be a nonzero element. We can write

$$x = x_1 \cdot \dots \cdot x_r, \quad x_1, \dots, x_r \text{ irreducible.}$$

Since \mathfrak{p} is prime, we have $x_i \in \mathfrak{p}$ for some i . We have thus constructed a nonzero irreducible element $y = x_i \in \mathfrak{p}$. We claim that it generates \mathfrak{p} .

Since A is a UFD, the ideal $\mathfrak{q} = (y)$ is prime: if $uv \in \mathfrak{q}$, i.e. $uv = wy$, unique factorization implies that y divides either u or v , so $u \in \mathfrak{q}$ or $v \in \mathfrak{q}$. Since $0 \neq \mathfrak{q} \subseteq \mathfrak{p}$ and \mathfrak{p} has height one, we must have $\mathfrak{q} = \mathfrak{p}$, and we are done. \square

Example 14.4.2 (Quadratic cone). The ring $A = k[x, y, z]/(xy - z^2)$ is an integrally closed domain of dimension 2 and the ideal $\mathfrak{p} = (x, z) \subseteq A$ (corresponding to a line passing through the vertex) is a prime of height one which is not principal. At the same time, the equality $xy = z^2$ exhibits the lack of unique factorization in A .

Proposition 14.4.3 (Stacks Project 031T). *Let A be an integrally closed noetherian domain and let K be its field of fractions. Then*

$$A = \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$$

(intersection of the localizations of A at all height one primes, taken inside K).

Proof. If $\dim(A) = 0$ then A is a field, and if $\dim(A) = 1$ then A is a discrete valuation ring and there is a unique height one prime \mathfrak{m} for which $A = A_{\mathfrak{m}}$. We might therefore assume that $\dim(A) \geq 2$.

The (\subseteq) inclusion is clear, so let us show (\supseteq) . Let $g/f \in K$ be a nonzero rational function ($f, g \in A \setminus \{0\}$) which belongs to $A_{\mathfrak{p}}$ for every height one prime $\mathfrak{p} \subseteq A$. We want to show that $g/f \in A$, or equivalently that $g \in (f)$, or that the image of g in $B = A/(f)$ is zero. Let us denote the maximal ideal of A by \mathfrak{m} and of B by \mathfrak{n} .

By Krull’s theorem all irreducible components of $\text{Spec}(B) = V(f) \subseteq \text{Spec}(A)$ have dimension $n - 1$, or equivalently for every minimal prime $\mathfrak{q} \subseteq B$ its preimage $\mathfrak{p} \subseteq A$ has height one. The assumption that

$f/g \in A_{\mathfrak{p}}$ implies that g maps to zero in $B_{\mathfrak{q}}$. In other words, the open subset $D(f) \cap V(f) \subseteq V(f) = \text{Spec}(B)$ contains all of the generic points of $\text{Spec}(B)$ and hence is dense in $\text{Spec}(B)$. We are therefore left with showing the following property of the ring B :

If $g \in B$ vanishes on a dense open subset of $\text{Spec}(B)$ then it is zero.

Suppose that $g \neq 0$ in B . Let $\mathfrak{q} \subseteq B$ be a minimal prime among the support $V(\text{Ann}_B(g)) \neq \emptyset$ of g , and let $\mathfrak{p} \subseteq A$ be its preimage. By our assumption, \mathfrak{q} is not a minimal prime of B , and hence $\dim(B_{\mathfrak{q}}) \geq 1$ and $\dim(A_{\mathfrak{p}}) \geq 2$. In order to obtain the desired contradiction with $g \neq 0$ in B , we may replace B with $B_{\mathfrak{q}}$ and A with $A_{\mathfrak{p}}$. In other words, we may assume that $\mathfrak{q} = \mathfrak{n}$ and $\mathfrak{p} = \mathfrak{m}$. Thus g is identically zero on $\text{Spec}(B)$ away from its closed point, and $\sqrt{\text{Ann}_B(g)} = \mathfrak{q}$.

Let $n \geq 1$ be the smallest such that $\mathfrak{n}^n \subseteq \text{Ann}_B(g)$ (i.e. $g \cdot \mathfrak{n}^n = 0$). Suppose first that $n = 1$, so $g \cdot \mathfrak{n} = 0$ in B , or $g \cdot \mathfrak{m} \subseteq (f)$ in A . There are two cases:

- If $g\mathfrak{m} = (f)$ in A , we have $\mathfrak{m} = (f/g)$ (as submodules of K), so \mathfrak{m} is principal, contradicting our assumption that $\dim(A) \geq 2$.
- Otherwise, we have $g\mathfrak{m} \subseteq f\mathfrak{m}$. In this case we have a well-defined endomorphism $g/f: \mathfrak{m} \rightarrow \mathfrak{m}$. By the Cayley–Hamilton trick (see Atiyah–Macdonald, Propositions 5.1 and 2.4) this implies that h/f is integral over A , and (since A is integrally closed) that $h/f \in A$, so $h \in (f)$, contradiction.

If $n > 1$, pick $h \in \mathfrak{n}^{n-1} \setminus \text{Ann}_B(g)$ and replace g with $g' = gh \neq 0$, so that $g' \cdot \mathfrak{n} = 0$, and run the previous argument. □

14.5. Problem session (Mar 17)

Before stating the problems we discussed (prompted by Yurii) the origin of the name “divisor.” This was a good opportunity to talk about some algebraic number theory.

Remark 14.5.1 (Why are divisors called *divisors*?). The terminology comes from algebraic number theory, in which problems with lack of unique factorization in rings of integers in number fields play a big role. Suppose for example that you want to prove that $x^p + y^p = z^p$ has no nontrivial solutions for a prime $p > 2$. Rewrite this as

$$x^p = z^p - y^p = \prod_{i=0}^{p-1} (z - \zeta_p^i y), \quad \zeta_p = \exp(2\pi i/p),$$

an equality in the ring $\mathbb{Z}[\zeta_p]$. Were this ring a UFD, the above equality would look at least unlikely.

Let K be a number field (a finite extension of \mathbb{Q}) and let $A = \mathcal{O}_K$ be its ring of integers (integral closure of \mathbb{Z}). Then A is a Dedekind domain (its local rings are DVRs), but not a PID/UFD in general (in dimension 1, UFD and PID mean the same thing, thanks to Proposition 14.4.1). A nonzero ideal $I \subseteq A$ can be written uniquely as a product of powers of prime ideals. Thus, if we work with ideals, we always have unique factorization (N.B. that’s why we call them *ideals* — for “ideal divisors”). Since they do not form a group, we work with *fractional* ideals, which are finitely generated \mathcal{O}_K -submodules of K . The **class group** $\text{Cl}(\mathcal{O}_K)$ is the quotient of the group of nonzero fractional ideals by the subgroup of principal fractional ideals. Thus $\text{Cl}(\mathcal{O}_K) = \text{Pic}(\text{Spec}(\mathcal{O}_K))$, and \mathcal{O}_K is a UFD if and only if $\text{Cl}(\mathcal{O}_K) = 0$. One of the first results in algebraic number theory (whose proof requires a bit of convex geometry) is that $\text{Cl}(\mathcal{O}_K)$ is always finite. Its order, called the **class number** of K , is a mysterious invariant which can be computed explicitly in special cases. For example, for the case of the cyclotomic extension $K = \mathbb{Q}(\zeta_p)$ we have $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$ and the class number is related to Bernoulli numbers. A prime p is **regular** if p does not divide the class number of $\mathbb{Q}(\zeta_p)$. Following ideas of Kummer, it is not difficult to prove Fermat’s Last Theorem for regular exponents p (and thus for all $p \leq 100$ except for $\{37, 59, 67\}$).

Exercise 14.5.2. Let \mathcal{L} be an invertible sheaf on a projective variety X such that $\Gamma(X, \mathcal{L})$ and $\Gamma(X, \mathcal{L}^{-1})$ are both nonzero. Show that $\mathcal{L} \simeq \mathcal{O}_X$.

Solution. We first check that every nonzero morphism $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ between invertible sheaves on a variety (or integral scheme) is injective. Since this is a local question, we may assume X is affine and \mathcal{L}_i both trivial. In this case the map can be viewed as a map $\mathcal{O}_X \rightarrow \mathcal{O}_X$, given by multiplication by a nonzero element $f \in \mathcal{O}(X)$. Since $\mathcal{O}(X)$ is a domain, this map is injective.

Now, we have

$$\Gamma(X, \mathcal{L}) = \text{Hom}(\mathcal{O}_X, \mathcal{L}), \quad \Gamma(X, \mathcal{L}^{-1}) = \text{Hom}(\mathcal{O}_X, \mathcal{L}^{-1}) = \text{Hom}(\mathcal{L}, \mathcal{O}_X)$$

(the last equality obtained by tensoring with \mathcal{L}). Thus by assumption we have nonzero maps $\mathcal{O}_X \rightarrow \mathcal{L}$ and $\mathcal{L} \rightarrow \mathcal{O}_X$. By the previous paragraph, these maps are injective. Further, the composition $\mathcal{O}_X \rightarrow \mathcal{L} \rightarrow \mathcal{O}_X$ corresponds to an element $f \in \mathcal{O}(X)$, which is nonzero since the composition is injective. But X is projective, so $\mathcal{O}(X) = k$. Thus $f \in k^\times$, and $\mathcal{O}_X \rightarrow \mathcal{L} \rightarrow \mathcal{O}_X$ is an isomorphism. Therefore the second map $\mathcal{L} \rightarrow \mathcal{O}_X$ is surjective, and since as already observed it is also injective, it is an isomorphism. \square

In the next problems we discuss smooth projective curves. For a divisor $D = \sum a_P P$ on a smooth projective curve X we define its **degree** as

$$\text{deg}(D) = \sum a_P \in \mathbb{Z}.$$

Exercise 14.5.3. Suppose $X = \mathbb{P}^1$. Show that for every principal divisor D we have $\text{deg}(D) = 0$.

Solution. Let $f \in k(X)^\times$ be a nonzero rational function, which we may write as p/q for a pair of polynomials $p, q \in k[T]$. It suffices to treat p and q separately, so we may assume that $f \in k[T]$ is a polynomial. In this case we have

$$\text{div}(f) = \sum_{x: f(x)=0} m_x \cdot x - m \cdot \infty$$

where the sum is over all zeros of f and m_x are their multiplicities. Thus $\sum m_x = \text{deg}(f)$. The multiplicity m at infinity is calculated as the order of the pole at zero of $f(1/T)$, which equals $\text{deg}(f)$. Thus $\text{deg div}(f) = 0$. \square

Remark 14.5.4. The result can also be deduced from our computation of $\text{Pic}(\mathbb{P}^1)$ in Lecture 14. Indeed, the maps $\text{deg}: \text{Div}(\mathbb{P}^1) \rightarrow \mathbb{Z}$ and $\pi: \text{Div}(\mathbb{P}^1) \rightarrow \text{Pic}(\mathbb{P}^1) = \mathbb{Z}$ are “the same” map.

Exercise 14.5.5. Same as the previous problem but without assuming $X = \mathbb{P}^1$.

This result allows us to define the degree of an invertible sheaf on a smooth projective curve:

$$\begin{array}{ccccccc} k(X)^\times & \longrightarrow & \text{Div}(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & 1 \\ & \searrow & \downarrow \text{deg} & & \uparrow \exists! \text{deg} & & \\ & 0 & \mathbb{Z} & & & & \end{array}$$

In order to attack Problem 14.5.5, we need two very useful facts.

Lemma 14.5.6. Let X be a normal (i.e. $\mathcal{O}(U)$ is integrally closed for every open affine $U \subseteq X$) variety with function field K and let L/K be a finite extension. Then there exists a unique normal variety Y with function field L and a finite dominant map $Y \rightarrow X$ inducing the field extension L/K .

Proof. Suppose first that X is affine and let $A = \mathcal{O}(X)$. In this case, let B be the integral closure of A in L . Then $Y = \text{MSpec}(B)$ is a normal variety with function field $\text{Frac}(B) = L$ and the induced $Y \rightarrow X$ is finite (by finiteness of integral closure) and dominant (as $A \rightarrow B$ is injective). The uniqueness is also clear: any

other candidate has to be of the form $\text{MSpec}(B')$ where B' is integrally closed, $\text{Frac}(B') = L$, and $A \rightarrow B'$ finite (and hence integral). Thus $B = B'$.

In order to globalize this, we check easily that this construction is compatible with localization: for every nonzero $f \in A$, the localization $B[f^{-1}]$ coincides with the integral closure of $A[f^{-1}]$ in L . This is clear since again $B[f^{-1}]$ is normal, has function field L , and $A[f^{-1}] \rightarrow B[f^{-1}]$ is finite.

For the general case, let $X = \bigcup_{\alpha \in I} U_\alpha$ be an affine open cover. For each $\alpha, \beta \in I$ let

$$U_\alpha \cap U_\beta = \bigcup_{\gamma \in J_{\alpha\beta}} U_{\alpha\beta\gamma}$$

be an open cover such that $U_{\alpha\beta\gamma}$ is a distinguished affine open in both U_α and U_β (see Problem 1 on Problem Set 4). Define $V_\alpha = \text{MSpec}(B_\alpha)$ where B_α is the integral closure of $A_\alpha = \mathcal{O}(U_\alpha)$ in L . The previous paragraph allows us to identify the preimage of each $U_{\alpha\beta\gamma}$ in V_α and V_β . This way we glue the V_α to obtain a normal variety Y with function field L together with a finite dominant map $Y \rightarrow X$, as desired. \square

Lemma 14.5.7. *Let X be a smooth curve, let $U \subseteq X$ be a dense open subset, and let $\phi: U \rightarrow \mathbb{P}^n$ be a map. Then ϕ extends uniquely to a map $\psi: X \rightarrow \mathbb{P}^n$.*

Proof. We have $U = X \setminus S$ for a finite set S . By induction it suffices to treat the case $\#S = 1$, i.e. $U = X \setminus \{x\}$. The ring $R = \mathcal{O}_{X,x}$ is a discrete valuation ring with fraction field $K = k(X)$. From this point on we can either work over $\text{Spec}(\mathcal{O}_{X,x})$ to extend the map locally at x , or (more clumsily) give a complete proof in the language of varieties. We go the second route.

Let $U_i \subseteq U$ be the preimages of the standard opens $D(x_i) \subseteq \mathbb{P}^n$. Since they cover the irreducible U , one of them has to be dense, say U_0 . Then $X_0 = U_0 \cup \{x\}$ is an open neighborhood of x (we are on a curve!). Since the problem is local we can replace X with X_0 . This means that ϕ can be represented by a map $(1 : f_1 : \dots : f_n)$ for some $f_1, \dots, f_n \in \mathcal{O}(U)$. Similarly, shrinking X around x we may assume that $x = V(\pi)$ for some $\pi \in \mathcal{O}(X)$, so that $\mathcal{O}(U) = \mathcal{O}(X)[1/\pi]$. If f_1, \dots, f_n belong to $\mathcal{O}(X) \subseteq \mathcal{O}(U)$, we extend the map with the same formula. In general (ignoring the indices for which $f_i = 0$), we can write $f_i = g_i/\pi^{m_i}$ where $g_i \in \mathcal{O}(X)$, $g_i \notin (\pi)$. Let $m = \min(m_1, \dots, m_n)$ and set $f'_0 = \pi^m$, $f'_i = \pi^m g_i$. Then one of the f'_i is a unit in a neighborhood of x , and after shrinking X again the map $(f'_0 : \dots : f'_n)$ gives the desired extension $\psi: X \rightarrow \mathbb{P}^n$. \square

Remark 14.5.8. The same ideas combined with Proposition 14.4.3 imply that a rational map from a normal variety to a projective variety is defined on an open subset whose complement has codimension at least two.

Corollary 14.5.9. *A birational map between smooth projective curves is an isomorphism.*

Corollary 14.5.10. *Let X be a smooth projective curve and let $f \in k(X)$ be a non-constant rational function. Then there exists a unique finite map $\phi: X \rightarrow \mathbb{P}^1$ (in practice denoted also by f) such that $\phi^*(t) = f$ (here T is the coordinate on $\mathbb{A}^1 \subseteq \mathbb{P}^1$, so that $k(\mathbb{P}^1) = k(t)$).*

Proof. Since f is non-constant, it is not algebraic over k , so there exists a unique k -algebra map of fields $k(t) \hookrightarrow k(X)$ sending t to f . So we have a finite extension of fields $k(X)/k(t)$, and by Lemma 14.5.6 there exists a unique finite dominant map $X' \rightarrow \mathbb{P}^1$ where X' is a normal variety and $k(X') = k(X)$. Since $\dim(X') = 1$, normal means smooth, so X' is a smooth projective curve. Since $k(X') = k(X)$, Lemma 14.5.7 implies that $X = X'$. \square

Lemma 14.5.11. *Let $\phi: Y \rightarrow X$ be a finite map between smooth curves. Let $d = \deg(k(Y)/k(X))$ be the degree of the field extension of function fields. Then for every $x \in X$ the number of points in $\phi^{-1}(x)$, counted with multiplicity, equals d .*

Proof. We may assume that X is affine. Let $A = \mathcal{O}_{X,x}$ and $B = \mathcal{O}(Y) \otimes_{\mathcal{O}(X)} A = \mathcal{O}(Y)_x$. Then $A \rightarrow B$ is finite and A is a dvr. Moreover, being a localization of $\mathcal{O}(Y)$, B is a domain (with fraction field $k(Y)$) and hence torsion-free as an A -module. By the classification of modules over PIDs we see that B is a free A -module of finite rank. By tensoring with $k(X)$, we see that this rank is equal to d . By tensoring with k , we obtain the sum of the multiplicities of all $y \mapsto x$. Indeed, if π is a uniformizer of A , then for every such y , the multiplicity is $v_y(\pi)$, which is the length of $\mathcal{O}_{Y,y}/(\pi)$, and we have

$$B \otimes_A k = \prod_{y \mapsto x} \mathcal{O}_{Y,y}/(\pi). \quad \square$$

15. Lecture 15 (Mar 19): Cohomology

Recommended reading: Kempf §8, Hartshorne III §2–3

15.1. Motivation

Let X be an algebraic set (or a noetherian scheme) and \mathcal{F} a coherent sheaf on X . In this lecture, we are going to associate to \mathcal{F} a sequence of vector spaces $H^q(X, \mathcal{F})$ (indexed by $q \geq 0$) called its *cohomology groups*. In brief, for $q = 0$, these are the global sections $\Gamma(X, \mathcal{F})$, and for a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$, the group $H^1(X, \mathcal{F}')$ allows us to control the failure of it staying exact after applying H^0 , then $H^2(X, \mathcal{F}')$ addresses a similar problem with H^1 , and so on.

Surprisingly, the actual definition of cohomology groups is almost unimportant, and it is challenging to give elements of $H^q(X, \mathcal{F})$ a concrete geometric meaning for $q \geq 2$ (for $q = 1$, one can describe the cohomology group in terms of torsors, see [SP Section 03AG]). What *is* important is how to compute them, and how to use their properties to prove new results in algebraic geometry.

What are the key properties of cohomology?

1. A short exact sequences of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

induces long exact sequences of cohomology groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(X, \mathcal{F}') & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}'') \\
 & & & & & & \downarrow \\
 & & & & & & H^1(X, \mathcal{F}') \longrightarrow \dots \\
 & & & & & & \downarrow \\
 & & & & & & \dots \longrightarrow H^{q-1}(X, \mathcal{F}'') \\
 & & & & & & \downarrow \\
 & & & & & & H^q(X, \mathcal{F}') \longrightarrow H^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}'') \\
 & & & & & & \downarrow \\
 & & & & & & H^{q+1}(X, \mathcal{F}') \longrightarrow \dots
 \end{array}$$

δ

2. If X is affine, then $H^q(X, \mathcal{F}) = 0$ for $q > 0$ (Theorem 15.3.1).
3. In general (assuming X is separated), $H^q(X, \mathcal{F})$ can be computed using the Čech complex associated to a fixed affine open cover of X .
4. We have $H^q(X, \mathcal{F}) = 0$ if X can be covered by q affine open subsets, or if $\dim(X) < q$.
5. If X is projective (or more generally complete), then $H^q(X, \mathcal{F})$ are finite-dimensional for all $q \geq 0$.

Up to 70% of core technical results in algebraic geometry rely on the computation or vanishing of some $H^q(X, \mathcal{O}_X(D))$. Here are two basic but important examples, a preview of things to come.

Example 15.1.1 (Connectedness of hyperplane sections). Let $X \subseteq \mathbb{P}^n$ be a normal projective variety of dimension ≥ 2 and let $D = X \cap H$ for a hyperplane $H \subseteq \mathbb{P}^n$. Then D is connected.

The proof of this is related to cohomology of line bundles as follows (since D might be non-reduced in general, it is quite essential to work with schemes for this proof). A projective scheme Y is connected if $\dim H^0(Y, \mathcal{O}_Y) = 1$ (if $Y = Y_0 \sqcup Y_1$ with $Y_i \neq \emptyset$, then considering locally constant functions we have

$k \times k \hookrightarrow H^0(Y, \mathcal{O}_Y)$; the converse holds if Y is reduced. So we want to compute $H^0(D, \mathcal{O}_D)$, which fits inside an exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

as follows:

$$k = H^0(X, \mathcal{O}_X) \longrightarrow H^0(D, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{O}_X(-D)).$$

Thus if $H^1(X, \mathcal{O}_X(-D)) = 0$, we deduce D is connected. This is not “if and only if,” and in fact $H^1(X, \mathcal{O}_X(-D))$ will often be nonzero. However, we are free to replace D with nD in this argument for any $n \geq 1$; the scheme $nD = V(\mathcal{O}_X(-nD))$ has the same underlying space as D , only more locally defined nilpotent functions. This time we get

$$k = H^0(X, \mathcal{O}_X) \longrightarrow H^0(nD, \mathcal{O}_{nD}) \longrightarrow H^1(X, \mathcal{O}_X(-nD)).$$

The proof is now reduced to showing that if X is normal and of dimension ≥ 2 , then $H^1(X, \mathcal{O}_X(-nD)) = 0$ for $n \gg 0$. See [Hartshorne, Corollary III 7.8].

Example 15.1.2 (Construction of rational functions on a curve). Let X be a smooth projective curve and let $P \in X$. Does there exist a rational function $f \in k(X)$ with a pole at P and regular on $U = X \setminus \{P\}$?

For $n \geq 0$, rational functions with a pole of order at most n at P , and regular everywhere else, are by definition the global sections of the invertible sheaf $\mathcal{O}_X(nP)$ on X introduced in Lecture 14. In the question, we seek a non-constant such function, which translates into the inequality

$$\dim H^0(X, \mathcal{O}_X(nP)) \geq 2.$$

The sheaves $\mathcal{O}_X(nP)$ and $\mathcal{O}_X((n+1)P)$ fit into a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(nP) \longrightarrow \mathcal{O}_X((n+1)P) \longrightarrow k_P \longrightarrow 0$$

where k_P is a one-dimensional skyscraper sheaf at the point P .

Cohomology enters the picture thanks to the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \mathcal{O}_X(nP)) & \longrightarrow & H^0(X, \mathcal{O}_X((n+1)P)) & \longrightarrow & k \\ & & & & & & \uparrow \\ & & & & & & \delta \\ & & & & & & \downarrow \\ & & & & & & \delta \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

(here the skyscraper sheaf k_P has global sections k and no higher cohomology). Thus, if $H^1(X, \mathcal{O}_X(nP)) = 0$ for some $n > 0$, we conclude that

$$\dim H^0(X, \mathcal{O}_X((n+1)P)) = 1 + \dim H^0(X, \mathcal{O}_X(nP)) \geq 2,$$

and that there exists a rational function f with pole of order exactly $(n+1)$ at P and regular on U .

The required vanishing is a consequence of the **Serre duality theorem**, which states that

$$H^1(X, \mathcal{O}_X(D)) \simeq \text{Hom}(\mathcal{O}_X(D), \omega_X)^\vee = H^0(X, \omega_X(-D))^\vee$$

where $\omega_X = \Omega_X^1$ is the **canonical invertible sheaf** (the sheaf of Kähler differentials, or of algebraic differential one-forms on X). In particular, if $\deg(D) > \deg(\omega_X)$, then $\text{Hom}(\mathcal{O}_X(D), \omega_X) = 0$, and consequently $H^1(X, \mathcal{O}_X(D)) = 0$. (As we shall see later as a consequence of the Riemann–Roch theorem, $\deg(\omega_X) = 2g - 2$ where

$$g = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X)$$

is the **genus** of X .) We conclude that for $n \gg 0$ (more precisely, for $n \geq 2g$) there exists a rational function f with pole of order exactly n at P and regular on U .

We can draw from this the following important corollary: the curve $U = X \setminus \{P\}$ is affine (and more generally, every proper open subset of X is affine). Indeed, reasoning as in the last problem session (see Lecture 14, §14.5) we see that the function f we have just constructed corresponds to a finite map

$$f: X \longrightarrow \mathbb{P}^1$$

with $f^{-1}(\infty) = \{P\}$. Thus $U = f^{-1}(\mathbb{P}^1 \setminus \{\infty\}) = f^{-1}(\mathbb{A}^1)$ is affine.

15.2. Definition and basic properties of sheaf cohomology

The following elementary exposition of sheaf cohomology follows Kempf §8.1.

Let \mathcal{F} be an abelian sheaf on a topological space X . In our discussion of sheafification in Lecture 7, we introduced the **sheaf of discontinuous sections**

$$D(\mathcal{F}) = \prod_{x \in X} i_{x,*} \mathcal{F}_x,$$

the product of skyscraper sheaves corresponding to the stalks of \mathcal{F} at all points of x . We have a tautological injective map

$$i: \mathcal{F} \longrightarrow D(\mathcal{F}).$$

The sheaf $D(\mathcal{F})$ is an example of a **flabby** (a.k.a. **flasque**) sheaf: one for which the restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ are surjective for all $V \subseteq U \subseteq X$. We recall the key property of flabby sheaves (see Problem Set 3):

- If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence with \mathcal{F}' flabby, then

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

Using the construction $D(\mathcal{F})$ iteratively, we can associate to every abelian sheaf \mathcal{F} a canonical **flabby resolution** $D^\bullet(\mathcal{F})$ (a.k.a. the *Godement resolution*): define $Q(\mathcal{F}) = D(\mathcal{F})/\mathcal{F}$ and set

$$D^0(\mathcal{F}) = D(\mathcal{F}), \quad D^1(\mathcal{F}) = D(Q(\mathcal{F})), \quad D^2(\mathcal{F}) = D(Q(Q(\mathcal{F}))), \quad \dots$$

and so on. Diagram for visual aid:

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \searrow & & & & \searrow \\
 & & & \mathcal{F} & & & & Q(Q(\mathcal{F})) & & \searrow \\
 & & & \parallel & & & & \nearrow & & 0 \\
 & & & \mathcal{F} & \xrightarrow{i} & D^0(\mathcal{F}) & \longrightarrow & D^1(\mathcal{F}) & \longrightarrow & \dots \\
 & & & \parallel & & \searrow & & \nearrow & & \\
 & & & \mathcal{F} & \longrightarrow & Q(\mathcal{F}) & & & & \\
 & & & & & \nearrow & & \searrow & & \\
 & & & & & 0 & & & & 0
 \end{array}$$

This being a resolution means that the middle row is exact. Applying the functor $\Gamma(X, -)$ to the complex of abelian sheaves $D^\bullet(\mathcal{F})$ we obtain a complex of abelian groups $\Gamma(X, D^\bullet(\mathcal{F}))$.

Definition 15.2.1. Let \mathcal{F} be an abelian sheaf on a topological space X . For $q \geq 0$, the q -th cohomology group of \mathcal{F} is

$$H^q(X, \mathcal{F}) = H^q(\Gamma(X, D^\bullet(\mathcal{F}))) = \frac{\ker(\Gamma(X, D^q(\mathcal{F})) \rightarrow \Gamma(X, D^{q+1}(\mathcal{F})))}{\text{im}(\Gamma(X, D^{q-1}(\mathcal{F})) \rightarrow \Gamma(X, D^q(\mathcal{F})))}.$$

We observe right away that left-exactness of $\Gamma(X, -)$ applied to $0 \rightarrow \mathcal{F} \rightarrow D^0(\mathcal{F}) \rightarrow D^1(\mathcal{F})$ gives

$$\Gamma(X, \mathcal{F}) \xrightarrow{\simeq} H^0(X, \mathcal{F}).$$

Note first of all that since $D(\mathcal{F})$ is obviously functorial in \mathcal{F} , so are all of the above constructions, in particular $H^q(X, \mathcal{F})$. Furthermore, if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of abelian sheaves, then so are

$$0 \longrightarrow D(\mathcal{F}') \longrightarrow D(\mathcal{F}) \longrightarrow D(\mathcal{F}'') \longrightarrow 0$$

and therefore also

$$0 \longrightarrow Q(\mathcal{F}') \longrightarrow Q(\mathcal{F}) \longrightarrow Q(\mathcal{F}'') \longrightarrow 0.$$

Iterating this, it follows that the sequence of complexes of sheaves

$$0 \longrightarrow D^\bullet(\mathcal{F}') \longrightarrow D^\bullet(\mathcal{F}) \longrightarrow D^\bullet(\mathcal{F}'') \longrightarrow 0$$

is exact. By our “key property” of flabby sheaves, it remains exact after applying $\Gamma(X, -)$, so we have a short exact sequence of complexes of abelian groups

$$0 \longrightarrow \Gamma(X, D^\bullet(\mathcal{F}')) \longrightarrow \Gamma(X, D^\bullet(\mathcal{F})) \longrightarrow \Gamma(X, D^\bullet(\mathcal{F}'')) \longrightarrow 0.$$

By the snake lemma, we obtain a long exact sequence of cohomology groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \mathcal{F}') & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}'') \\ & & \dots & & \dots & & \dots \\ & \longrightarrow & H^{q-1}(X, \mathcal{F}') & \longrightarrow & H^{q-1}(X, \mathcal{F}) & \longrightarrow & H^{q-1}(X, \mathcal{F}'') \\ & & \searrow & & \searrow & & \searrow \\ & & & & & & \delta \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \\ & & H^q(X, \mathcal{F}') & \longrightarrow & H^q(X, \mathcal{F}) & \longrightarrow & H^q(X, \mathcal{F}'') \\ & & \searrow & & \searrow & & \searrow \\ & & & & & & \delta \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \\ & & H^{q+1}(X, \mathcal{F}') & \longrightarrow & H^{q+1}(X, \mathcal{F}) & \longrightarrow & H^{q+1}(X, \mathcal{F}'') \longrightarrow \dots \end{array}$$

Lemma 15.2.2. (a) If \mathcal{F} is flabby then $H^q(X, \mathcal{F}) = 0$ for $q > 0$.

(b) If $\mathcal{F} \rightarrow \mathcal{C}^\bullet$ is a flabby resolution (meaning that $0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow \dots$ is exact and all the \mathcal{C}^i are flabby), then we have a canonical isomorphism

$$H^q(X, \mathcal{F}) = H^q(\Gamma(X, \mathcal{C}^\bullet)).$$

(c) More generally, (b) holds under the weaker (thanks to (a)) assumption that

$$H^q(X, \mathcal{C}^p) = 0 \quad \text{for } q > 0 \text{ and all } p \geq 0.$$

Proof. (a) We first check the following auxiliary fact: if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence with \mathcal{F}' and \mathcal{F} flabby, then \mathcal{F}'' is flabby as well. To show this, take $V \subseteq U \subseteq X$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) & \longrightarrow & 0. \end{array}$$

Here the rows are exact since \mathcal{F}' is flabby (by our “key fact” applied to the restriction of the sequence to U or V), and the left and middle vertical maps are surjective since \mathcal{F}' and \mathcal{F} are flabby. It follows that the right map is surjective (by the snake lemma or an easy diagram chase), and since U and V were arbitrary, we see that \mathcal{F}'' is flabby.

Now, the auxiliary fact implies that $Q(\mathcal{F})$ is flabby. By induction, all $Q^i(\mathcal{F})$ are flabby, and the “key fact” implies that our big “visual aid” diagram remains exact after applying $\Gamma(X, -)$. Thus $\Gamma(X, D^\bullet(\mathcal{F}))$ is exact in positive degrees, and hence $H^q(X, \mathcal{F}) = 0$ for $q > 0$.

(c) Induction on $q \geq 0$. This is true for $q = 0$ (with no assumptions on the \mathcal{C}^p). Let \mathcal{G} be the cokernel of $\mathcal{F} \rightarrow \mathcal{C}^0$ so that $\mathcal{G} \rightarrow \mathcal{C}^{\bullet \geq 1}$ gives a resolution of \mathcal{G} of the same kind, and the cohomology sequence of the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

gives

$$H^1(X, \mathcal{F}) = \text{cok}(H^0(X, \mathcal{C}^0) \rightarrow H^0(X, \mathcal{G})) = H^1(\Gamma(X, \mathcal{C}^\bullet), \quad H^q(X, \mathcal{F}) \simeq H^{q-1}(X, \mathcal{G}) \quad (q \geq 2),$$

so the result for $(\mathcal{G}, q-1)$ implies the result for (\mathcal{F}, q) . \square

Lemma 15.2.3. *Let X be a noetherian topological space and let $\{\mathcal{F}_\alpha\}$ be an inductive system of abelian sheaves on X , and let $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$ be its direct limit. Then*

$$\varinjlim H^q(X, \mathcal{F}_\alpha) \xrightarrow{\sim} H^q(X, \mathcal{F})$$

for all $q \geq 0$.

Proof. We have (hopefully) shown this for $q = 0$. The general case is left as a (worthwhile) exercise. \square

Remark 15.2.4. More generally, the result holds on any spectral space (we do not need *all* opens to be quasi-compact, but only a basis of quasi-compact opens, stable under intersections).

Exercise 15.2.5. Compute $H^1(S^1, \underline{\mathbb{Z}})$ directly from our definition of cohomology.

Remark 15.2.6 (Functoriality). Trivially, $H^q(X, \mathcal{F})$ is functorial in \mathcal{F} . It is also functorial in X , in the following way(s). Let $f: Y \rightarrow X$ be a map and let \mathcal{F} be an abelian sheaf on Y . Then we have natural maps

$$f^*: H^q(X, f_*\mathcal{F}) \longrightarrow H^q(Y, \mathcal{F})$$

defined as follows (more abstract definitions of cohomology yield easier constructions). For every flabby sheaf \mathcal{G} on Y , the push-forward $f_*\mathcal{G}$ is flabby. It follows that we have a complex on X

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*D^0(\mathcal{F}) \longrightarrow f_*D^1(\mathcal{F}) \longrightarrow \dots$$

whose terms $f_*D^p(\mathcal{F})$ are flabby for $p > 0$, and which is exact on the left (but not exact in general). For each $p \geq 0$ and $q \geq 1$ we obtain maps

$$H^q(X, f_*Q^p) \xrightarrow{\sim} H^{q-1}(X, f_*D^q(\mathcal{F})/f_*Q^p) \longrightarrow H^{q-1}(X, f_*Q^{p+1})$$

where the first map is the boundary map from the short exact sequence $0 \rightarrow f_*\mathcal{Q}^p \rightarrow f_*D^p(\mathcal{F}) \rightarrow f_*D^p(\mathcal{F})/f_*\mathcal{Q}^p \rightarrow 0$ and the second map is induced by the canonical injection $f_*D^p(\mathcal{F})/f_*\mathcal{Q}^p \hookrightarrow f_*\mathcal{Q}^{p+1}$. For $q = 0$, we instead get a map

$$H^1(X, \pi_*\mathcal{Q}^p) \rightarrow \text{cok}(H^0(X, f_*D^p(\mathcal{F})) \rightarrow H^0(X, f_*\mathcal{Q}^{p+1})) = \text{cok}(H^0(Y, D^p(\mathcal{F})) \rightarrow H^0(Y, \mathcal{Q}^{p+1})) = H^q(Y, \mathcal{F}).$$

We define the pull-back map $f^* : H^q(X, f_*\mathcal{F}) \rightarrow H^q(Y, \mathcal{F})$ as the composition

$$H^q(X, f_*\mathcal{F}) = H^q(X, f_*\mathcal{Q}^0) \rightarrow H^{q-1}(X, f_*\mathcal{Q}^1) \rightarrow \cdots \rightarrow H^1(X, f_*\mathcal{Q}^{q-1}) \rightarrow H^q(Y, \mathcal{F}).$$

The following variant is often used. Suppose now that f is a map of ringed spaces and let \mathcal{F} be an \mathcal{O}_X -module. Setting $\mathcal{G} = f^*\mathcal{F}$, we have the adjunction map $\mathcal{F} \rightarrow f_*f^*\mathcal{F} = f_*\mathcal{G}$. We define the pull-back map

$$f^* : H^q(X, \mathcal{F}) \longrightarrow H^q(Y, f^*\mathcal{F})$$

as the composition

$$H^q(X, \mathcal{F}) \xrightarrow{\text{adj.}} H^q(X, f_*f^*\mathcal{F}) \xrightarrow{f^* \text{ for } f^*\mathcal{F}} H^q(Y, f^*\mathcal{F}).$$

15.3. Cohomology of an affine

Theorem 15.3.1. *Let \mathcal{F} be a quasi-coherent sheaf on an affine algebraic set⁷ X . Then*

$$H^q(X, \mathcal{F}) = 0 \quad \text{for } q > 0.$$

Proof. This can be deduced from the exactness of the Čech complex (Lecture 9, §9.1) using some Čech cohomology techniques. Grothendieck does this in EGA III, using a theorem of Cartan (Gode-ment *Geometrie algébrique et théorie des faisceaux*, II 5.9.2). The Stacks Project also does this [SP Lemma 01XB], with a slightly easier approach (avoiding the use of spectral sequences, see [SP Lemma 01EW]). See also [Kempf §8.2], [Hartshorne Theorem III 3.5]. □

Remark 15.3.2. Theorem 15.3.1 allows us to “rediscover” the fact (proved in Lecture 11) that if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence of sheaves on an affine X with \mathcal{F}' a quasi-coherent \mathcal{O}_X -module, then

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact. Indeed, this is a consequence of the vanishing $H^1(X, \mathcal{F}') = 0$.

Remark 15.3.3. Serre proved the following converse to Theorem 15.3.1 (see [Hartshorne Theorem III 3.7]): suppose that X is an algebraic set or an affine noetherian scheme such that $H^1(X, \mathcal{J}) = 0$ for every coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. Then X is affine.

Sketch of proof: The assumption implies that for every closed subscheme/subset $Z \subseteq X$, the restriction $\mathcal{O}(X) \rightarrow \mathcal{O}(Z)$ is surjective. If $U \subseteq X$ is an open subset and $x \in U$, applying this to $Z = (X \setminus U) \cup \{x\}$ and the function $\chi_{\{x\}}$ (1 on $\{x\}$, 0 on $X \setminus U$) we find an $f \in \mathcal{O}(X)$ such that $x \in D(f) \subseteq U$. Thus $D(f)$'s form a base of the topology on X . Let $X = \bigcup U_\alpha$ be an affine open cover. We may find $f_1, \dots, f_r \in \mathcal{O}(X)$ (finitely many since X is qc) such that $X = D(f_1) \cup \cdots \cup D(f_r)$ and for every i there exists an α such that $D(f_i) \subseteq U_\alpha$. Now $D(f_i) = \text{Spec}(\mathcal{O}(U_\alpha)[f_i^{-1}])$ are affine. Thus, by the (corrected) Problem 3 on Problem Set 4, to show that X is affine it suffices to prove that f_1, \dots, f_r generate the unit ideal in $\mathcal{O}(X)$. To this end, we look at the short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}_X^r \xrightarrow{(f_1, \dots, f_r)} \mathcal{O}_X \longrightarrow 0;$$

⁷The same proof works for a noetherian scheme. The result is true for non-noetherian schemes as well.

the map on the right is surjective since the $D(f_i)$ cover X , so that on $D(f_i)$ the sequence has a splitting, and \mathcal{R} (the “sheaf of relations” between the f_i) is defined as the kernel. If $H^1(X, \mathcal{R}) = 0$, the map $(f_1, \dots, f_r): \mathcal{O}(X)^r \rightarrow \mathcal{O}(X)$ is surjective, so $1 = \sum g_i f_i$ for some $g_i \in \mathcal{O}(X)$ and we win.

To show that $H^1(X, \mathcal{R}) = 0$ we define $\mathcal{R}_i = \mathcal{R} \cap \mathcal{O}_X^i$ for $i = 0, \dots, r$. We have $\mathcal{R} = \mathcal{R}_r$, $\mathcal{R}_{i-1} \subseteq \mathcal{R}_i$, and the quotients $\mathcal{R}_i/\mathcal{R}_{i-1}$ embed into \mathcal{O}_X , so that $H^1(X, \mathcal{R}_i/\mathcal{R}_{i-1}) = 0$. The cohomology sequences associated to the short exact sequences

$$0 \longrightarrow \mathcal{R}_{i-1} \longrightarrow \mathcal{R}_i \longrightarrow \mathcal{R}_i/\mathcal{R}_{i-1} \longrightarrow 0$$

allow us to show by induction on i that $H^1(X, \mathcal{R}_i) = 0$ for all i .

Corollary 15.3.4. *Let $f: Y \rightarrow X$ be an affine map and let \mathcal{F} be a quasi-coherent sheaf on Y . Then the maps*

$$f^*: H^q(X, f_*\mathcal{F}) \longrightarrow H^q(Y, \mathcal{F})$$

are isomorphisms for all $q \geq 0$.

Example 15.3.5. Examples of affine maps to keep in mind:

- (a) closed immersions;
- (b) the inclusion of an affine open $j: U \hookrightarrow X$ into a separated X ;
- (c) the projection $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$.

16. Lecture 16 (Apr 2): Cohomology of coherent sheaves on projective varieties

Recommended reading: Hartshorne III.4 and III.5, Kempf §9.1–9.3

16.1. Čech cohomology

We shall now use the vanishing of cohomology of quasi-coherent sheaves on affines to give an algorithm for computing cohomology on an arbitrary separated algebraic set.

Let X be a separated algebraic set (or noetherian scheme), \mathcal{F} a quasi-coherent \mathcal{O}_X -module, and $U \subseteq X$ an affine open subset. Denote by $j: U \rightarrow X$ the inclusion map, which is an affine morphism: the preimage $j^{-1}(V) = U \cap V$ of every affine open $V \subseteq X$ is affine. Consequently, the maps

$$j^*: H^q(X, j_*(\mathcal{F}|_U)) \longrightarrow H^q(U, \mathcal{F}|_U)$$

are isomorphisms, and hence $H^q(X, j_*(\mathcal{F}|_U)) = 0$ for $q > 0$. The restriction maps $\mathcal{F}(V) \rightarrow \mathcal{F}(U \cap V)$ for varying $V \subseteq X$ assemble to give a canonical “restriction” map $\mathcal{F} \rightarrow j_*(\mathcal{F}|_U)$ towards a sheaf with no higher cohomology. This observation is the basis for the construction of the Čech complex of \mathcal{F} .

Construction. Let $X = U_0 \cup \dots \cup U_r$ be a finite affine open cover. For a non-empty $I \subseteq \{0, \dots, r\}$ denote by $U_I = \bigcap_{i \in I} U_i$ the corresponding intersection (which again is affine since X is separated), by $j_I: U_I \rightarrow X$ the inclusion map, and let

$$\mathcal{F}_I = j_{I,*}(\mathcal{F}|_{U_I}) = j_{I,*}j_I^*\mathcal{F}.$$

Consider the following complex of quasi-coherent sheaves on X :

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{|I|=1} \mathcal{F}_I \longrightarrow \bigoplus_{|I|=2} \mathcal{F}_I \longrightarrow \dots \quad (16.1.1)$$

where the map $\mathcal{F} \rightarrow \bigoplus_{|I|=1} \mathcal{F}_I$ is the sum of the restriction maps $\mathcal{F} \rightarrow \mathcal{F}_I$, and where the differential

$$d: \bigoplus_{|I|=p} \mathcal{F}_I \longrightarrow \bigoplus_{|J|=p+1} \mathcal{F}_J$$

is given by

$$d(f_I)_J = \sum_{j \in J} (-1)^{s(j)} f_{I \setminus \{j\}}.$$

Here $s(j) = \#\{j' \in J : j' < j\}$, so if $J = \{j_0 < \dots < j_p\}$ then $s(j_m) = m$.

Lemma 16.1.1. *The complex of sheaves (16.1.1) is exact.*

Proof. Not to obscure a simple picture with overwhelming notation, consider first the case $r = 1$, so $X = U_0 \cup U_1$. Let $x \in X$, it suffices to show exactness on stalks at x . Without loss of generality, we may assume that $x \in U_0$. In this case $(\mathcal{F}_0)_x = \mathcal{F}_x$ and a moment’s thought shows that $(\mathcal{F}_{01})_x = (\mathcal{F}_1)_x$ (as U_1 and U_{01} are equal in a neighborhood of x , namely on U_0). The stalks of the Čech complex at x thus takes the form

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x \oplus (\mathcal{F}_1)_x \longrightarrow (\mathcal{F}_1)_x \longrightarrow 0.$$

This is visibly (split) exact.

For the general case, let again $x \in X$, and assume $x \in U_0$. Then for every $I \subseteq \{0, \dots, r\}$ the opens U_I and $U_{I \cup \{0\}}$ are equal in a neighborhood of x , and hence

$$(\mathcal{F}_I)_x = (\mathcal{F}_{I \cup \{0\}})_x.$$

If we set $C_p = \bigoplus_{I \subseteq \{1, \dots, r\}, |I|=p+1} (\mathcal{F}_I)_x$, then we can decompose the stalk at x of the p -th term of the Čech complex of \mathcal{F} as follows

$$\bigoplus_{|I|=p+1} (\mathcal{F}_I)_x = \bigoplus_{0 \in I} (\mathcal{F}_I)_x \oplus \bigoplus_{I \subseteq \{1, \dots, r\}} (\mathcal{F}_I)_x = C_p \oplus C_{p-1}.$$

The stalk of the complex (16.1.1) at x takes the following form

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x \oplus C_0 \longrightarrow C_0 \oplus C_1 \longrightarrow C_1 \oplus C_2 \longrightarrow \dots$$

which again is split exact. \square

Definition 16.1.2. Let X be a separated algebraic set (or noetherian scheme), let $X = U_0 \cup \dots \cup U_r$ be a finite affine open cover, and let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. The **Čech complex** of \mathcal{F} relative to the cover $\{U_i\}_{i=0}^r$ is the complex

$$\mathcal{C}^\bullet(\mathcal{F}, \{U_i\}) = \left[\bigoplus_{|I|=1} \mathcal{F}(U_I) \longrightarrow \bigoplus_{|I|=2} \mathcal{F}(U_I) \longrightarrow \dots \right]$$

obtained by taking the global sections of the complex (16.1.1).

Corollary 16.1.3. Let X be a separated algebraic set (or noetherian scheme) and let $X = U_0 \cup \dots \cup U_r$ be a finite affine open cover. Then for every quasi-coherent \mathcal{O}_X -module \mathcal{F} on X we have

$$H^q(X, \mathcal{F}) = H^q(\mathcal{C}^\bullet(\mathcal{F}, \{U_i\})).$$

Proof. Since each U_I is affine, we have $H^q(X, \bigoplus_{|I|=p} \mathcal{F}_I) = 0$ for $q > 0$ and all $p \geq 0$ (see the paragraph preceding our construction of the Čech complex). We conclude applying Lemma 16.1.1 and Lecture 15, Lemma 15.2.2(c). \square

Corollary 16.1.4. Let X be a separated scheme which can be covered by $r+1$ affine open subsets. Then $H^q(X, \mathcal{F}) = 0$ for $q > r$ and every quasi-coherent sheaf \mathcal{F} on X .

Proof. This follows from Corollary 16.1.3 since $\mathcal{C}^\bullet(\mathcal{F}, \{U_i\})$ is concentrated in degrees $[0, r]$. \square

Example 16.1.5 (Punctured plane). Let $X = \mathbb{A}^2 \setminus \{(0, 0)\}$ with coordinates x_0, x_1 , covered by $U_i = D(x_i)$. Let us compute $H^q(X, \mathcal{O}_X)$ using the associated Čech complex

$$\begin{array}{ccc} [\mathcal{O}(U_0) \oplus \mathcal{O}(U_1) & \xrightarrow{d} & \mathcal{O}(U_{01}) \\ \parallel & & \parallel \\ k[x_0^{\pm 1}, x_1] \oplus k[x_0, x_1^{\pm 1}] & \xrightarrow{(f_0, f_1) \mapsto f_0 - f_1} & k[x_0^{\pm 1}, x_1^{\pm 1}]. \end{array}$$

The kernel of the bottom map is $k[x_0, x_1]$ embedded diagonally (and we knew already that $\mathcal{O}(X) = k[x_0, x_1]$), and the cokernel is spanned by monomials $x_0^{n_0} x_1^{n_1}$ with both exponents strictly negative, which we can write as

$$H^1(X, \mathcal{O}_X) = x_0^{-1} x_1^{-1} k[x_0^{-1}, x_1^{-1}].$$

The groups $H^q(X, \mathcal{O}_X)$ are zero for $q > 1$.

We mention the following result without proof.

Theorem 16.1.6 (Grothendieck vanishing, [Hartshorne III 2.7]). Let X be a noetherian topological space of dimension n . Then $H^q(X, \mathcal{F}) = 0$ for $q > n$ and every abelian sheaf \mathcal{F} on X .

16.2. Cohomology of projective space

Our next goal is to compute the groups

$$H^q(\mathbb{P}^n, \mathcal{O}(d))$$

for all $q \geq 0$, $n \geq 1$, and $d \in \mathbb{Z}$. As we shall see later (Theorem 16.2.5), every coherent sheaf on \mathbb{P}^n is a quotient of a finite direct sum of some $\mathcal{O}(d_i)$'s, and so our computation will have some fairly general consequences. We start our calculation with the structure sheaf on $\mathbb{A}^{n+1} \setminus \{0\}$, which is only a bit harder than the one for $n = 1$ done in Example 16.1.5.

Lemma 16.2.1. *For $n \geq 1$ we have*

$$H^q(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}) = \begin{cases} k[x_0, \dots, x_n] & q = 0 \\ 0 & 0 < q < n \\ (x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_n^{-1}] & q = n \\ 0 & q > n. \end{cases}$$

Proof. The proof is by induction on n . We have shown this for $n = 1$ in Example 16.1.5. Suppose that $n > 1$ and that the result holds for $n - 1$. Let $U = U_n = D(x_n)$ and let $j: U \rightarrow X$ be the inclusion. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow j_* \mathcal{O}_U \longrightarrow \mathcal{Q} \longrightarrow 0 \quad (16.2.1)$$

where \mathcal{Q} is supported on the closed subset $X' = V(x_n) \simeq \mathbb{A}^n \setminus \{0\}$. Even though it is not a $\mathcal{O}_{X'}$ -module, it decomposes as a direct sum of $\mathcal{O}_{X'}$ -modules

$$\mathcal{Q} = \bigoplus_{m < 0} \mathcal{O}_{X'} \cdot x_n^m.$$

Since X' is noetherian, cohomology commutes with infinite direct sums, so by the induction assumption we obtain

$$H^q(X, \mathcal{Q}) = H^q(X', \mathcal{Q}) = \begin{cases} x_n^{-1} k[x_0, \dots, x_{n-1}][x_n^{-1}] & q = 0 \\ 0 & 0 < q < n - 1 \\ (x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_{n-1}^{-1}, x_n^{-1}] & q = n - 1 \\ 0 & q > n - 1. \end{cases}$$

Since U is affine, $H^q(X, j_* \mathcal{O}_U) = H^q(U, \mathcal{O}_U) = 0$. Thus the long cohomology exact sequence associated to (16.2.1) gives

$$H^1(X, \mathcal{O}_X) = \text{cok}(\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(X, \mathcal{Q})) = \text{cok}(k[x_0, \dots, x_n, x_n^{-1}] \rightarrow x_n^{-1} k[x_0, \dots, x_n^{-1}]),$$

which is zero, and $H^q(X, \mathcal{O}_X) \simeq \bigoplus_{m < 0} H^{q-1}(X', \mathcal{O}'_X) \cdot x_n^m$. Thus we conclude by induction. \square

Lemma 16.2.2. *Let $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the projection. Then*

$$\pi_* \mathcal{O} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d).$$

Proof. We already know that for an open $U \subseteq \mathbb{P}^n$, we have $\mathcal{O}(d)(U) = \mathcal{O}(\pi^{-1}(U))_d$, the space of all functions on $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ which are homogeneous of degree d . So we need to show that every function on $\pi^{-1}(U) \subseteq \mathbb{A}^{n+1}$ can be written (uniquely) as a sum of homogeneous functions. Uniqueness is checked easily, and then existence can be checked locally. We can therefore reduce to the case $U = D(g)$ for some homogeneous g , in which case $\mathcal{O}(U) = k[x_0, \dots, x_n][g^{-1}]$. An element of this ring is a unique sum of f/g^m for homogeneous f of degree e , and such an element gives a homogeneous function of degree $e - md$. \square

Theorem 16.2.3. Let $n \geq 1$, $d \in \mathbb{Z}$, $q \geq 0$. Then

$$H^q(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} k[x_0, \dots, x_n]_d & q = 0 \\ 0 & 0 < q < n \\ ((x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_n^{-1}])_d \simeq (x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_n^{-1}]_{d+n+1} & q = n \\ 0 & q > n. \end{cases}$$

Proof. Since the map $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ is affine, by Lemma 16.2.2 we have

$$H^q(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}) = H^q(\mathbb{P}^n, \pi_* \mathcal{O}) = H^q(\mathbb{P}^n, \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(d)) = \bigoplus_{d \in \mathbb{Z}} H^q(\mathbb{P}^n, \mathcal{O}(d)).$$

On the other hand, the left-hand side is given in Lemma 16.2.1. Moreover, for $q = 0$ the degree d part of the right-hand side corresponds to $k[x_0, \dots, x_n]_d$ on the left (this is again Lemma 16.2.2). For $q = n$, a closer inspection of the Čech complex computation of $H^n(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O})$ (which we omit) shows that the degree d part of the right-hand side corresponds to the degree d part of $(x_0 \dots x_n)^{-1} k[x_0^{-1}, \dots, x_n^{-1}]$. \square

Corollary 16.2.4. The spaces $H^q(\mathbb{P}^n, \mathcal{O}(d))$ are finite-dimensional. Their dimensions are given as

$$H^q(\mathbb{P}^n, \mathcal{O}(d)) = \begin{cases} \binom{n+d}{n} & q = 0, d \geq 0 \\ 0 & q = 0, d < 0 \\ 0 & 0 < q < n \\ \binom{-d-1}{n} & q = n, d < 0 \\ 0 & q > n. \end{cases}$$

The following result is one of the most important tools. We omit the proof (it is not so difficult given what we know).

Theorem 16.2.5 (Serre, [Hartshorne II 5.17 and 5.18]). Let \mathcal{F} be a coherent sheaf on \mathbb{P}^n . Then there exists an integer d , an $N \geq 0$, and a surjection

$$\mathcal{O}(-d)^N \longrightarrow \mathcal{F}.$$

Remark 16.2.6. The following equivalent form of Theorem 16.2.5 is often given. Let us call \mathcal{F} **globally generated** if the natural map

$$\Gamma(\mathbb{P}^n, \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{F}$$

is surjective. (Note for example that if \mathcal{F} is an invertible sheaf this simply means: for every $x \in \mathbb{P}^n$ there exists a global section of \mathcal{F} which does not vanish at x .) Then Theorem 16.2.5 is equivalent to the statement that for $d \gg 0$, the coherent sheaf $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}(d)$ is globally generated.

Corollary 16.2.7. Let X be a projective algebraic set and let \mathcal{F} be a coherent sheaf on \mathbb{P}^n . Then the cohomology groups $H^q(X, \mathcal{F})$ are finite-dimensional for all $q \geq 0$. In particular, $\Gamma(X, \mathcal{F})$ is finite-dimensional.

Proof. Let $i: X \rightarrow \mathbb{P}^n$ be a closed immersion into some projective space. Since i is finite, the sheaf $i_* \mathcal{F}$ is coherent and $H^q(X, \mathcal{F}) \simeq H^q(\mathbb{P}^n, i_* \mathcal{F})$. Thus we may assume that $X = \mathbb{P}^n$.

We prove the result by *descending* induction on q . For $q > n$ we have $H^q(\mathbb{P}^n, \mathcal{F}) = 0$ since \mathbb{P}^n can be covered by $n + 1$ affines. For the induction step, let $q > 0$ and suppose the result has been proved for all $q' > q$. By Theorem 16.2.5 we have a surjection $\mathcal{O}(-d)^N \longrightarrow \mathcal{F}$, and letting \mathcal{R} be its kernel, a short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}(-d)^N \longrightarrow \mathcal{F} \longrightarrow 0.$$

The relevant portion of the long cohomology exact sequence is

$$H^q(\mathbb{P}^n, \mathcal{O}(-d))^N \longrightarrow H^q(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^{q+1}(\mathbb{P}^n, \mathcal{R})$$

where the group on the left is finite-dimensional by Corollary 16.2.4 and the one on the right is finite-dimensional by induction assumption. Thus $H^q(X, \mathcal{F})$ is finite-dimensional. \square

17. Lecture 17 (Apr 7): Differentials

Recommended reading: Hartshorne II 8, Kempf §6.3

17.1. Review of Kähler differentials

To a map of rings $A \rightarrow B$ we associate the B -module $\Omega_{B/A}^1$ of **Kähler differentials** of B over A . It is the target of an A -linear derivation

$$d: B \longrightarrow \Omega_{B/A}^1$$

(i.e. an A -module map into a B -module satisfying the Leibniz rule $d(fg) = f \cdot d(g) + g \cdot d(f)$) which is universal in the sense that every A -linear derivation $\delta: B \rightarrow M$ into a B -module M factors as $\delta = \bar{\delta} \circ d$ for a unique B -module map $\bar{\delta}: \Omega_{B/A}^1 \rightarrow M$. For completely formal reasons, $\Omega_{B/A}^1$ is generated as a B -module by the elements df ($f \in B$) subject to the relations

$$d(f+g) = df + dg, \quad d(ff') = fdg + gdf, \quad da = 0 \quad (f, f' \in B, a \in A).$$

Applying these inductively, we deduce that for every polynomial $f \in A[T]$ and $b \in B$, we have $d(f(b)) = f'(b)db$ where f' is the formal derivative of f .

If $B = A[T_1, \dots, T_n]/(f_1, \dots, f_r)$ is a (finite) presentation of B as an A -algebra, then $\Omega_{B/A}^1$ is generated by the finite number of elements dT_i subject to the relations

$$0 = df_j = \sum_{i=1}^n \frac{\partial f_j}{\partial T_i} dT_i,$$

or equivalently can be written as the cokernel of the ‘‘Jacobian matrix’’ $J = [\partial f_j / \partial T_i]$ treated as a map $B^n \rightarrow B^n$.

The following proposition lists the most important properties of $\Omega_{B/A}^1$. It is phrased in the less-than-optimal way, postulating the existence of certain canonical maps instead of constructing them. See the subsequent remark for extra precision.

Proposition 17.1.1. *Let $A \rightarrow B$ be a map of rings.*

(a) *(Base change) Let $A \rightarrow A'$ be a map of rings and let $B' = B \otimes_A A'$. Then*

$$\Omega_{B'/A'}^1 \simeq \Omega_{B/A}^1 \otimes_B B'.$$

(b) *(Transitivity) For every map of rings $B \rightarrow C$ we have an exact sequence of C -modules*

$$\Omega_{B/A}^1 \otimes_A C \longrightarrow \Omega_{C/A}^1 \longrightarrow \Omega_{C/B}^1 \longrightarrow 0.$$

(c) *(Localization) For every multiplicative system $S \subseteq B$, we have*

$$\Omega_{B[S^{-1}]/A}^1 \simeq \Omega_{B/A}^1[S^{-1}].$$

(d) *(Conormal sequence) For every ideal $I \subseteq B$ we have an exact sequence of B/I -modules*

$$I/I^2 \xrightarrow{d} \Omega_{B/A}^1 \otimes_B B/I \longrightarrow \Omega_{(B/I)/A}^1 \longrightarrow 0.$$

(e) *(Characterization in terms of the diagonal) Let J be the kernel of the surjective map $B \otimes_A B \rightarrow B$, $b \otimes b' \mapsto bb'$. Then*

$$\Omega_{B/A}^1 \simeq J/J^2.$$

Remark 17.1.2. Construction of the various maps appearing above:

- (a) Since $B \rightarrow B' \rightarrow \Omega_{B'/A}^1$ is a B -linear derivation, by the universal property of $\Omega_{B/A}^1$ we obtain a B -linear map $\Omega_{B/A}^1 \rightarrow \Omega_{B'/A}^1$, which by adjunction between forgetful and base change corresponds to a B' -linear map $\Omega_{B/A}^1 \otimes_B B' \rightarrow \Omega_{B'/A}^1$.
- (b) The first map in the sequence is adjoint to the A -module map $\Omega_{B/A}^1 \rightarrow \Omega_{C/A}^1$ corresponding to the A -linear derivation $B \rightarrow C \rightarrow \Omega_{C/A}^1$. The second one corresponds to the A -linear derivation $C \rightarrow \Omega_{C/B}^1$.
- (c) To get the map $\Omega_{B/A}^1[S^{-1}] \rightarrow \Omega_{B[S^{-1}]/A}^1$, apply the construction in (b) to $C = B[S^{-1}]$.
- (d) The second map is again (b), this time applied to $C = B/I$. The first one is the unique map making the diagram below commute

$$\begin{array}{ccc} I & \xrightarrow{d|_I} & \Omega_{B/A}^1 \\ \downarrow & & \downarrow \\ I/I^2 & \dashrightarrow & \Omega_{B/A}^1 \otimes_B B/I \end{array}$$

(since the left map is surjective, the desired map is unique if it exists. One still needs to check that it does.)

- (e) The morphism $\Omega_{B/A}^1 \rightarrow J/J^2$ is the unique B -module map sending db to the class of $b \otimes 1 - 1 \otimes b$.

Proof of Proposition 17.1.1. See Stacks Project: (a) [SP Lemma 00RV] (b) [SP Lemma 00RS] (c) [SP Lemma 00RT] (d) [SP Lemma 00RU] (e) [SP Lemma 00RW]. \square

17.2. Sheaves of differentials

Let X be an algebraic set. We can construct a coherent \mathcal{O}_X -module Ω_X^1 together with a k -linear derivation

$$d: \mathcal{O}_X \longrightarrow \Omega_X^1$$

in either of the following ways:

- (a) as the universal k -linear derivation into a *quasi-coherent* \mathcal{O}_X -module;
- (b) as the sheaf whose value on an affine open $U \subseteq X$ is $\Omega_{\mathcal{O}(U)/k}^1$ (this works thanks to Proposition 17.1.1(c));
- (c) as $\mathcal{J}_\Delta/\mathcal{J}_\Delta^2$ where $\mathcal{J}_\Delta \subseteq \mathcal{O}_{X \times X}$ is the ideal sheaf of the diagonal $X \simeq \Delta \subseteq X \times X$ (assuming that X is separated).

We omit the details — see the recommended references or Stacks Project, section 01UM.

Remark 17.2.1. In basically the same way, we define $\Omega_{X/S}^1$ for a map of schemes $X \rightarrow S$. The resulting sheaf is always quasi-coherent, and is coherent if S is locally noetherian and $X \rightarrow S$ is locally of finite type.

Proposition 17.2.2. *A variety X is smooth if and only if Ω_X^1 is locally free.*

Proof. The key point of the argument is that the fiber $\Omega_X^1(x)$ at a point $x \in X$ is isomorphic to the Zariski cotangent space ${}_x\mathcal{O}_X^2$ of the local ring $\mathcal{O}_{X,x}$. Assuming this, X is smooth iff the dimension of ${}_x\mathcal{O}_X^2 = \Omega_X^1(x)$ is constant, which happens precisely when Ω_X^1 is locally free. To see the claim (which we've already seen in some other form), we note that for a k -vector space V , considered as an $\mathcal{O}_{X,x}$ -module via $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x = k$, k -linear maps $\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow V$ correspond to k -linear derivations $\mathcal{O}_{X,x} \rightarrow V$. Thus the k -vector spaces $\Omega_X^1(x)$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ represent the same functor and hence are isomorphic. \square

Our first (non-affine) example is the projective space.

Theorem 17.2.3 (Euler sequence). *On $X = \mathbb{P}^n$, there is a short exact sequence of coherent sheaves*

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \longrightarrow \mathcal{O}(-1)^{n+1} \xrightarrow{(x_0, \dots, x_n)} \mathcal{O} \longrightarrow 0.$$

Proof. The key point behind the construction of the left map in the sequence is that for a homogeneous f of degree d , the partial derivatives $\partial f / \partial x_i$ are homogeneous of degree $d - 1$. For $i = 0, \dots, n$ we define a k -linear derivation $\partial_i: \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)$ as follows. For an open $U \subseteq \mathbb{P}^n$ of the form $U = D(f)$ for a homogeneous $f \in k[x_0, \dots, x_n]$ of degree d , we have

$$\mathcal{O}(U) = k[x_0, \dots, x_n][f^{-1}]_0$$

and we define $\partial_i(g/f^m)$ (where $g \in k[x_0, \dots, x_n]_{md}$) by the standard formula

$$\partial_i \left(\frac{g}{f^m} \right) = \frac{\partial(g/f^m)}{\partial x_i} = \frac{\left(\frac{\partial g}{\partial x_i} \right) f - mg \left(\frac{\partial f}{\partial x_i} \right)}{f^m},$$

which is a homogeneous element of $k[x_0, \dots, x_n][f^{-1}]$ of degree -1 , i.e. an element of $\Gamma(U, \mathcal{O}_{\mathbb{P}^n}(-1))$. It is trivial to check that these maps glue compatibly and that they give rise to a k -linear derivation.

Taken together, the maps ∂_i give a k -linear derivation $\mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$, and hence a map of coherent sheaves

$$\Omega_{\mathbb{P}^n}^1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}.$$

In order to check the exactness of the resulting sequence, it suffices to check on the standard opens. On $U_0 = D(x_0)$, the sequence takes the form

$$\bigoplus_{i=1}^n k[y_1, \dots, y_n] dy_i \longrightarrow k[y_1, \dots, y_n] e_0 \oplus \bigoplus_{i=1}^n k[y_1, \dots, y_n] e_i \longrightarrow k[y_1, \dots, y_n]$$

where the first map sends dy_i to $e_i - x_i e_0$ and the second sends $e_0 \mapsto 1$ and $e_i \mapsto x_i$ for $i > 0$. The image of the first and the image of the second are both equal to the set of $f_0 e_0 + \sum_{i=1}^n f_i e_i$ satisfying $f_0 = -\sum_{i=1}^n x_i f_i$. The rest is clear. \square

Proposition 17.2.4 (Conormal sequence). *Let X be a smooth variety and let $Y \subseteq X$ be a smooth divisor (codimension one subvariety). Then $\mathcal{J}_Y \simeq \mathcal{O}_X(-Y)$, and we have a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-Y)|_Y \longrightarrow \Omega_X^1|_Y \longrightarrow \Omega_Y^1 \longrightarrow 0.$$

Proof. Without injectivity on the left, this is the global version of the conormal sequence for Kähler differentials (see Proposition 17.1.1(d)). In order to check that the left map is injective, it is enough to check it on fibers at all $y \in Y$. Let $f \in \mathcal{O}_{X,y}$ be a local generator of \mathcal{J}_Y . Then $f \in \mathfrak{m}_y \setminus \mathfrak{m}_y^2$, and hence its image in $\Omega_X^1|_Y(y) = \Omega_X^1(y) = \mathfrak{m}_y / \mathfrak{m}_y^2$ is nonzero. \square

17.3. The canonical bundle

Definition 17.3.1. Let X be a smooth algebraic variety.

- (a) For $p \geq 0$ we define Ω_X^p to be the p -th exterior power of Ω_X^1 . (By convention, $\Omega_X^0 = \mathcal{O}_X$.) We call Ω_X^p the sheaf of **differential p -forms**.
- (b) We call the dual $\mathcal{H}om(\Omega_X^1, \mathcal{O}_X) = (\Omega_X^1)^\vee$ the **tangent sheaf** of X and denote it by \mathcal{T}_X .
- (c) Suppose that $\dim(X) = n$. Then $\omega_X = \Omega_X^n$ is an invertible sheaf, called the **canonical sheaf** of X .

In the special case when X is a curve, we have $\omega_X = \Omega_X^1$.

The computation of ω_X using various short exact sequences is enabled by the following linear algebra fact. For a locally free \mathcal{O}_X -module \mathcal{E} on a ringed space X , let us define $\det(\mathcal{E}) = \bigwedge^r(\mathcal{E})$ where $r = \text{rk}(\mathcal{E})$. Thus $\omega_X = \det(\Omega_X^1)$.

Lemma 17.3.2. *Let $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ be a sequence of locally free sheaves on a ringed space X . Then*

$$\det(\mathcal{F}) \simeq \det(\mathcal{E}) \otimes \det(\mathcal{G}).$$

Proof. If X is a point and $\mathcal{O}(X)$ is a field, this lemma postulates an isomorphism between two one-dimensional vector spaces. If we want to globalize this, we need a canonical such isomorphism which would also work for free modules over a ring. Intuitively, the assertion is that choosing a volume form on a vector space is equivalent to choosing one on a subspace and on the quotient space, a Fubini-type result.

If u_1, \dots, u_e is a basis of \mathcal{E} and w_1, \dots, w_g a basis of \mathcal{G} , then $u = u_1 \wedge \dots \wedge u_e$ and $w = w_1 \wedge \dots \wedge w_g$ are basis elements of $\det(\mathcal{E})$ and $\det(\mathcal{G})$, respectively. Further, if $v_1, \dots, v_g \in \mathcal{F}$ are elements lifting w_1, \dots, w_g , then $u_1, \dots, u_e, v_1, \dots, v_g$ is a basis of \mathcal{F} and $v = u_1 \wedge \dots \wedge u_e \wedge v_1 \wedge \dots \wedge v_g$ is a basis of $\det(\mathcal{F})$. The postulated isomorphism maps $u \otimes w$ to v . We check that different choices lead to the same isomorphism. If u'_1, \dots, u'_e and w'_1, \dots, w'_g are different choices of bases, then $u' = \det(A)u$ and $w' = \det(B)w$ where A and B are the respective change of base matrices. At the same time, we have $v' = \det(C)v$ where C is a change of base matrix from $u_1, \dots, u_e, v_1, \dots, v_g$ to $u'_1, \dots, u'_e, v'_1, \dots, v'_g$ for some lifts v'_1, \dots, v'_g of w'_1, \dots, w'_g . The matrix C has a block-triangular form

$$C = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix}$$

and hence $\det(C) = \det(A) \cdot \det(B)$. The exact same calculation works for a short exact sequence of free modules over a commutative ring.

For the globalization step, consider the diagram

$$\begin{array}{ccc} & \bigwedge^e \mathcal{E} \otimes \bigwedge^g \mathcal{F} & \\ & \swarrow^{1 \otimes q} & \searrow \\ \bigwedge^e \mathcal{E} \otimes \bigwedge^g \mathcal{G} & & \bigwedge^{e+g} \mathcal{F} \\ \parallel & & \parallel \\ \det(\mathcal{E}) \otimes \det(\mathcal{G}) & \cdots \cdots \cdots \rightarrow & \det(\mathcal{F}). \end{array}$$

The left arrow is the identity of $\bigwedge^e \mathcal{E}$ tensored with the surjection $\bigwedge^g \mathcal{F} \rightarrow \bigwedge^g \mathcal{G}$ and hence is surjective. The right arrow is given by exterior multiplication. We claim that there exists a unique dotted arrow making the diagram commute, and that it is an isomorphism. Uniqueness is clear (since the left map is surjective), and then existence (and being an isomorphism) can be checked locally. We may therefore assume that \mathcal{E} , \mathcal{F} , and \mathcal{G} are free, in which case we can use the preceding discussion. \square

Example 17.3.3 (The canonical sheaf of \mathbb{P}^n). Let us compute $\omega_{\mathbb{P}^n}$ using the Euler sequence. Lemma 17.3.2 immediately yields

$$\omega_{\mathbb{P}^n} = \det(\Omega_{\mathbb{P}^n}^1) = \det(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}) \otimes \det(\mathcal{O}_{\mathbb{P}^n})^{-1} = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Example 17.3.4 (Adjunction formula). Let Y be a smooth divisor on a smooth variety X . Then the conormal sequence gives

$$\omega_Y = \det(\Omega_Y^1) = \det(\Omega_X^1|_Y) \otimes \det(\mathcal{O}_X(-Y)|_Y)^{-1} = \omega_X|_Y \otimes \mathcal{O}_X(Y)|_Y = \omega_X(Y)|_Y.$$

For example, for a degree d plane curve $Y \subseteq \mathbb{P}^2$, the canonical sheaf is

$$\omega_Y \simeq \mathcal{O}_{\mathbb{P}^2}(d-3)|_Y.$$

For $d = 3$ this gives $\omega_Y \simeq \mathcal{O}_Y$.

17.4. Serre duality

For now, we simply state the Serre duality theorem for curves.

Theorem 17.4.1 (Serre duality). *Let X be a smooth projective curve.*

(a) *There is a canonical isomorphism $H^1(X, \omega_X) \simeq k$.*

(b) *For every invertible sheaf \mathcal{L} on X , the bilinear map*

$$H^0(X, \mathcal{L}^\vee \otimes \omega_X) \times H^1(X, \mathcal{L}) = \text{Hom}(\mathcal{L}, \omega_X) \times H^1(X, \mathcal{L}) \longrightarrow H^1(X, \omega_X) \simeq k$$

(defined by functoriality of $H^1(X, -)$) is a perfect pairing, i.e. it gives an isomorphism

$$H^1(X, \mathcal{L}) \simeq H^0(X, \mathcal{L}^{-1} \otimes \omega_X)^\vee.$$

Definition 17.4.2. Let X be a smooth projective curve. We define its **genus** as the number

$$g(X) = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X).$$

Example 17.4.3. For a plane curve $Y \subseteq \mathbb{P}^2$ of degree d , we have $g(Y) = (d-1)(d-2)/2$.

17.5. Problem session (Apr 7)

During the problem session:

1. We discussed **Serre's theorem** (every coherent sheaf on \mathbb{P}^n is a quotient of $\mathcal{O}(-d)^m$ for some d and m) and how it implies that $H^q(X, \mathcal{F})$ is finite dimensional for a coherent sheaf \mathcal{F} on a projective variety X . (This is covered in detail in the last section of Lecture 16.) We mentioned the fact (**Hilbert's syzygy theorem**) that every \mathcal{F} on \mathbb{P}^n admits a *finite* resolution by direct sums of $\mathcal{O}(d_i)$'s.
2. This let us to define the **Euler characteristic** of a coherent sheaf \mathcal{F} on a projective variety X as

$$\chi(\mathcal{F}) = \sum_{q=0}^{\infty} (-1)^q \dim H^q(X, \mathcal{F}).$$

We showed that if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence of coherent sheaves then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Using the formulas from Lecture 16, we calculated $\chi(\mathcal{O}_{\mathbb{P}^n}(d))$ explicitly as

$$\chi(\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d} = \frac{(d+1)(d+2)\dots(d+n)}{n!}.$$

We deduced from this and from the syzygy theorem that $\chi(\mathcal{F}(d))$ is a polynomial in d (for any coherent sheaf \mathcal{F} on \mathbb{P}^n). Here $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$.

3. By adapting the proof of the finite dimensionality in (1) (namely, descending induction on q) we showed that for a coherent sheaf \mathcal{F} on \mathbb{P}^n , the groups $H^q(\mathbb{P}^n, \mathcal{F}(d))$ are zero for $q > 0$ and $d \gg 0$.
4. We defined Ω_X^q , \mathcal{T}_X , and ω_X (see Definition 17.3.1) and covered the rest of §17.3.

18. Lecture 18 (Apr 9): Riemann–Roch

Recommended reading: Hartshorne IV 1, Kempf §7–8

18.1. Basic facts about curves

Let C be a smooth and complete one-dimensional variety and let $k(C)$ be its function field.

- (1) Every rational map from C to a projective (or just complete) variety X is everywhere defined.
- (2) We have $k(C) = \text{Hom}(C, \mathbb{P}^1)$.
- (3) Every map $f: C \rightarrow C'$ to another smooth complete curve is either constant or finite. In the latter case, $f_*\mathcal{O}_C$ is locally free of rank $d = \deg(f) = [k(C) : k(C')]$, and for every divisor D on C' we have $\deg(f^*D) = \deg(f) \cdot \deg(D)$.
- (4) The curve C is projective (Sketch of proof: a non-constant $f \in k(C)$ gives a finite map $f: C \rightarrow \mathbb{P}^1$. Let $L = f^*\mathcal{O}(1)$ and consider the graded ring $R = \bigoplus_{n \geq 0} H^0(C, L^n) = \bigoplus_{n \geq 0} H^0(\mathbb{P}^1, (f_*\mathcal{O}_C)(n))$. We can show it is finitely generated, and so $R = k[x_0, \dots, x_n]/I$, $C = V(I) \subseteq \mathbb{P}^n$.)
- (5) For a non-constant $f \in k(C)$ corresponding to a finite map $f: C \rightarrow \mathbb{P}^1$, we have $\text{div}(f) = f^*((0) - (\infty))$.
- (6) The degree of every principal divisor on C is zero. (This follows from (3) and (5).)

18.2. The Riemann–Roch theorem

Let C be a smooth projective curve. We introduce the following notation and terminology:

- $\ell(D) = \dim H^0(C, \mathcal{O}_C(D)) = \dim\{f \in k(C) : \text{div}(f) \geq -D\}$;
- K is any divisor for which $\mathcal{O}_C(K) \simeq \omega_X$ (a **canonical divisor**);
- $g = \ell(K_C)$ is the **genus** of C ;
- for a coherent sheaf \mathcal{F} on C we write $\chi(\mathcal{F}) = \dim H^0(C, \mathcal{F}) - \dim H^1(C, \mathcal{F})$ for its **Euler characteristic**.

Theorem 18.2.1. *Let C be a smooth projective curve and D a divisor on C . Then*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

Proof. Denote the left-hand side by $\text{LHS}(D)$ and the right-hand side by $\text{RHS}(D)$. By Serre duality:

$$\begin{aligned} \text{LHS}(D) &= \dim H^0(C, \mathcal{O}_C(D)) - \dim H^0(X, \mathcal{O}_C(D)^\vee \otimes \omega_C) \\ &= \dim H^0(C, \mathcal{O}_C(D)) - \dim H^1(C, \mathcal{O}_C(D))^\vee \\ &= \chi(\mathcal{O}_C(D)). \end{aligned}$$

For a point $P \in C$ consider $D' = D + P$. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D') \longrightarrow k_P \longrightarrow 0$$

where k_P is the skyscraper sheaf at P . From this we get

$$\text{LHS}(D') = \chi(\mathcal{O}_C(D')) = \chi(\mathcal{O}_C(D)) + \chi(k_P) = \text{LHS}(D) + 1,$$

and at the same time obviously

$$\text{RHS}(D') = \text{RHS}(D) + 1.$$

Thus the assertion holds for D if and only if it holds for D' . By induction, it suffices to verify the formula for $D = 0$:

$$\text{LHS}(0) = \ell(0) - \ell(K) = 1 - g = \text{RHS}(0). \quad \square$$

Corollary 18.2.2. *We have $\deg(\omega_C) = 2g - 2$.*

Corollary 18.2.3. *We have $\ell(D) \geq \deg(D) + 1 - g$, with equality if $\deg(D) > 2g - 2$.*

Corollary 18.2.4. *Every proper open subset $U \subseteq X$ is affine.*

Proof. Consider first the case $U = X \setminus \{P\}$ for a single point P . Then $\ell(nP) > 1$ for $n \gg 0$, so we can find a non-constant rational function $f \in k(C)$ with $\text{div}(f) \geq -nP$. Thus f has a pole at P and no other poles, and hence the corresponding finite map $f: C \rightarrow \mathbb{P}^1$ satisfies $f^{-1}(\infty) = \{P\}$, so $U = f^{-1}(\mathbb{A}^1)$ is affine.

For the general case, write $U = X \setminus \{P_1, \dots, P_r\}$, and for each $i = 1, \dots, r$ let f_i be a rational function constructed above for $P = P_i$. Take $f = f_1 + \dots + f_r$, then f has poles precisely at P_1, \dots, P_r , so again for the corresponding map $f: C \rightarrow \mathbb{P}^1$ we have $U = f^{-1}(\mathbb{A}^1)$. \square

18.3. Curves of genus zero (a.k.a. rational curves)

Lemma 18.3.1. *Let C be a smooth projective curve. The following are equivalent:*

- (a) $g(C) = 0$;
- (b) there exist $P, Q \in C$ with $P \neq Q$ and $P \sim Q$ (meaning that $P - Q$ is a nonzero principal divisor);
- (c) $C \simeq \mathbb{P}^1$.

Proof. (a) \Rightarrow (b): By Riemann–Roch, we have $\ell(P - Q) = 1$, so that $P - Q \sim D$ for an effective divisor D . But $\deg(D) = 0$, so $D = 0$.

(b) \Rightarrow (c): Suppose that $P - Q = \text{div}(f)$ and let $f: C \rightarrow \mathbb{P}^1$ be the corresponding finite map. Then $Q = f^*(\infty)$ and hence $\deg(f) = 1$, so that f is an isomorphism.

(c) \Rightarrow (a): We computed that $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, so $g = \ell(K) = 0$. \square

18.4. Curves of genus one (a.k.a. elliptic curves)

Definition 18.4.1. An **elliptic curve** is a pair $(E, \underline{0})$ where E is a smooth projective curve of genus one and $\underline{0} \in E$ is a point.

Lemma 18.4.2. *Let E be an elliptic curve. We have $\omega_E \simeq \mathcal{O}_E$.*

Proof. We have $\deg(\omega_E) = 2g - 2 = 0$ and $\dim H^0(E, \omega_E) = g = 1$, so that $\omega_E \simeq \mathcal{O}_E$. \square

Lemma 18.4.3. *Let $\text{Pic}^0(E) \subseteq \text{Pic}(E)$ be the subgroup consisting of divisor classes of degree zero. The map $E \rightarrow \text{Pic}^0(E)$ mapping $P \in E$ to the class of $P - \underline{0}$ is bijective.*

Proof. Injectivity follows from Lemma 18.3.1. For surjectivity, let $D = \sum n_i P_i$ be a divisor of degree zero, which we rewrite as

$$D = \sum n_i (P_i - \underline{0}) + (\sum n_i) \underline{0} = \sum n_i (P_i - \underline{0}).$$

By induction on the number of terms, it suffices to show that for every $P, Q \in E$ there exists a (unique) $R \in E$ such that

$$(P - \underline{0}) + (Q - \underline{0}) \sim (R - \underline{0}).$$

By Riemann–Roch, we have $\ell(D) = \deg(D)$ as long as $\deg(D) > 0$. Apply this to $D = P + Q - \underline{0}$ to get $\ell(P + Q - \underline{0}) = 1$. Thus there exists a unique effective divisor R with $P + Q - \underline{0} \sim R$. Since $\deg(R) = \deg(P + Q - \underline{0}) = 1$, this divisor R is a point, and we are done. \square

Corollary 18.4.4. *There is a natural structure of a commutative group variety on E with neutral element $\underline{0}$.*

(Technically we did not show that the group structure $E \times E \rightarrow E$ is a morphism of varieties.)

More facts about elliptic curves (assuming k is not of characteristic 2 or 3): There exists a finite map $E \rightarrow \mathbb{P}^1$ of degree two, ramified at four points, which after a change of coordinates on \mathbb{P}^1 can be taken to be $0, 1, \infty, \lambda$. The curve is then isomorphic to the plane curve

$$y^2z = x(x-z)(x-\lambda z),$$

where the point $\underline{0}$ is sent to $(0 : 1 : 0)$. For three points $P, Q, R \in E$, we have $P + Q + R = 0$ in the group structure on E if and only if there exists a line $\ell \subseteq \mathbb{P}^2$ such that $\ell \cap \mathbb{P}^2 = P + Q + R$ as divisors.

18.5. Bonus: proof of Serre duality

We shall outline Kempf’s ingenious proof of Serre duality for curves. As we have seen in both the proof of Riemann–Roch and in its applications, important information about an invertible sheaf \mathcal{L} on a smooth projective curve C is encoded by the boundary map

$$\delta_P: k \longrightarrow H^1(C, \mathcal{L})$$

induced by the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(P) \longrightarrow k_P \longrightarrow 0$$

for a chosen point $P \in C$: the map δ_P is zero if and only if there is a section of $\mathcal{L}(P)$ which is not a section of \mathcal{L} . The idea behind the proof is to consider all of these maps at the same time as the point P varies. As we shall see, taken together they assemble into a map of vector bundles on C .

In order to state this precisely, we first need an important correction: the quotient $\mathcal{L}(P)/\mathcal{L}$ is only non-canonically isomorphic to the skyscraper k_P at P ; rather, it is the skyscraper at P with value the fiber of $\mathcal{L}(P)$ at P^8 . For example, for $\mathcal{L} = \mathcal{O}_C$, this quotient is $\mathfrak{m}_P^{-1}/\mathcal{O}_{C,P}$; this space is dual to $\mathfrak{m}_P/\mathfrak{m}_P^2$, which is the fiber of ω_C at P . In general, it will be this tensored with \mathcal{L} . So it makes sense to expect that there exists a map of locally free sheaves on C of the form

$$\delta: \mathcal{L} \otimes \omega_C^{-1} \longrightarrow H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C;$$

whose fiber at P is the map δ_P . Here the source can be identified with $\mathcal{H}om(\omega_C, \mathcal{L})$ and the target is the free sheaf with fiber $H^1(C, \mathcal{L})$. In particular, if $\mathcal{L} = \omega_C$, the map δ produces an element τ of

$$H^0(C, H^1(C, \omega_C) \otimes_k \mathcal{O}_C) = H^0(C, \mathcal{O}_C) \otimes_k H^1(C, \omega_C) = H^1(C, \omega_C).$$

This turns out to be a basis element (i.e. the map δ for $\mathcal{L} = \omega_C$ is an isomorphism), yielding the isomorphism $H^1(C, \omega_C) \xrightarrow{\sim} k$ which is part of the formulation of Serre duality.

We relegate the definition of the map δ to the next subsection (see Proposition 18.6.1). Here we show how to complete the proof of Serre duality assuming its existence.

We begin with some preliminaries about cohomology of invertible sheaves on curves.

⁸Warning: by accident, this fiber should be denoted by $\mathcal{L}(P)(P)$. Since we are on a curve, a point can be considered a divisor, and so two notational conventions are in conflict: $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{x,x}} \kappa(x)$ for the fiber of a coherent sheaf \mathcal{F} at a point $x \in X$, and $\mathcal{L}(D) = \mathcal{L} \otimes_{\mathcal{O}_X}(D)$ for an invertible sheaf \mathcal{L} and a divisor D on X .

Lemma 18.5.1. *Let \mathcal{L} be an invertible sheaf on a smooth projective curve C , and let D be a divisor on C for which $\mathcal{L} \simeq \mathcal{O}_X(D)$. Having made this choice, we regard \mathcal{L} as a subsheaf of the constant sheaf \mathcal{K}_X .*

(a) *The exact sequence*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{K}_C \longrightarrow \mathcal{K}_C/\mathcal{L} \longrightarrow 0 \quad (18.5.1)$$

is a flabby resolution of \mathcal{L} .

(b) *Consequently, we have*

$$H^1(C, \mathcal{L}) \simeq \text{coker}(k(C) \rightarrow \Gamma(C, \mathcal{K}_C/\mathcal{L})).$$

(c) *The group $H^1(C, \mathcal{L})$ is zero if and only if, for every effective divisor E and every $P \in C$ we have*

$$\dim \Gamma(C, \mathcal{L}(E+P)) = \dim \Gamma(C, \mathcal{L}(E)) + 1. \quad (18.5.2)$$

Proof. (a) The sheaf \mathcal{K}_C is the constant sheaf with value $k(C)$, and hence is flabby as C is irreducible. On the other hand, we can write

$$\mathcal{K}_C = \varinjlim_{E \geq 0} \mathcal{L}(E),$$

and hence $\mathcal{K}_C/\mathcal{L} = \varinjlim_{E \geq 0} \mathcal{L}(E)/\mathcal{L}$ (by exactness of direct limits). Now each quotient sheaf $\mathcal{L}(E)/\mathcal{L}$ is a direct sum of skyscraper sheaves, and hence is flabby. Since C is noetherian, taking sections commutes with direct limits of sheaves, and we conclude that $\mathcal{K}_C/\mathcal{L}$ is flabby. In fact, it is isomorphic to the direct sum of the skyscraper sheaves $k(C)/\mathcal{L}_P$ at all $P \in C$.

(b) Follows from (a) by Lecture 15, Lemma 15.2.2(b).

(c) The condition is equivalent to the claim that

$$\dim \Gamma(C, \mathcal{L}(E)) = \dim \Gamma(C, \mathcal{L}) + \deg E \quad \text{for all } E \geq 0. \quad (18.5.3)$$

The exact sequence (18.5.1) is the inductive limit of the sequences

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{L} \longrightarrow 0. \quad (18.5.4)$$

Our is thus equivalent to the statement that these sequences remain exact after taking global sections. Taking inductive limits is exact, so condition (18.5.3) implies that the exact sequence (18.5.1) is exact after applying global sections, which means precisely that $H^1(C, \mathcal{L}) = 0$. Conversely, if $H^1(C, \mathcal{L}) = 0$, then the cohomology exact sequence associated to (18.5.4) shows that $\Gamma(C, \mathcal{L}(E)) \rightarrow \Gamma(C, \mathcal{L}(E)/\mathcal{L})$ is surjective, as desired. \square

Lemma 18.5.2. *Let P_1, \dots, P_r be distinct points on C and let \mathcal{L} be an invertible sheaf. Then the boundary map $k^r \rightarrow H^1(C, \mathcal{L})$ induced by the sequence*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(P_1 + \dots + P_r) \longrightarrow \bigoplus_{i=1}^r k_{P_i} \longrightarrow 0 \quad (18.5.5)$$

equals the direct sum of the maps $\delta_{P_1}, \dots, \delta_{P_r}$.

Proof. For every effective divisor E , we can regard $\Gamma(C, \mathcal{L}(E)/\mathcal{L})$ as a subspace of $\Gamma(C, \mathcal{K}_C/\mathcal{L})$, and then $\delta_E: \Gamma(C, \mathcal{L}(E)/\mathcal{L}) \rightarrow H^1(C, \mathcal{L})$ is the restriction of the surjection $\delta: \Gamma(C, \mathcal{K}_C/\mathcal{L}) \rightarrow H^1(C, \mathcal{L})$ to that subspace. In the situation at hand, if $E = P_1 + \dots + P_r$, the subspace $\Gamma(C, \mathcal{L}(E)/\mathcal{L}) \subseteq \Gamma(C, \mathcal{K}_C/\mathcal{L})$ is the direct sum of the one-dimensional subspaces $\Gamma(C, \mathcal{L}(P_i)/\mathcal{L})$ for $i = 1, \dots, r$. \square

The very existence of the map δ has an important consequence.

Corollary 18.5.3. *If $\deg(\mathcal{L}) > \deg(\omega_C)$ then $H^1(C, \mathcal{L}) = 0$.*

Proof. The assumption implies that $\mathcal{L} \otimes \omega_C^{-1}$ has positive degree, and hence there are no nonzero maps $\mathcal{L} \otimes \omega_C^{-1} \rightarrow \mathcal{O}_C$. Consequently, the map δ for \mathcal{L} is zero (as the target is non-canonically the direct sum of copies of \mathcal{O}_C). Therefore $\delta(P) = \delta_P$ is zero for all $P \in C$. The same argument applies to $\mathcal{L}(D)$ for any effective divisor D . By Lemma 18.5.1(c), this implies that $H^1(C, \mathcal{L}) = 0$. \square

Corollary 18.5.4. *The map δ for $\mathcal{L} = \omega_C$ gives an isomorphism $\mathcal{O}_C = \omega_C \otimes \omega_C^{-1} \xrightarrow{\sim} H^1(C, \omega_C) \otimes_k \mathcal{O}_C$. Consequently, $H^1(C, \omega_C)$ is canonically isomorphic to k .*

Proof. Consider the set of integers d for which there exists an invertible sheaf \mathcal{L} of degree d with $H^1(C, \mathcal{L}) \neq 0$. By Corollary 18.5.3, the set is also bounded from above. We can therefore find an invertible sheaf \mathcal{L} with $H^1(C, \mathcal{L}) \neq 0$ such that $\deg(\mathcal{L})$ is maximal (if $H^1(C, \mathcal{L}) = 0$ for all \mathcal{L} (a case which never happens and is easily ruled out), take any \mathcal{L} in what follows). Then $H^1(C, \mathcal{L}(P)) = 0$ for every $P \in C$, and consequently $\delta_P \neq 0$ for every P . It follows that the map $\delta: \mathcal{L} \otimes \omega_C^{-1} \rightarrow H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$ is an isomorphism: the invertible sheaf $\mathcal{L} \otimes \omega_C^{-1}$ is isomorphic to the free sheaf $H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$. Then $\mathcal{L} \simeq \omega_C$, $H^1(C, \mathcal{L})$ is one-dimensional, and (taking $\mathcal{L} = \omega_C$) that we have a canonical isomorphism $H^1(C, \omega_C) \simeq k$. \square

Since $H^0(C, \mathcal{O}_C) = k$, for any two vector spaces V and W we have $\text{Hom}(V, W) \xrightarrow{\sim} \text{Hom}(V \otimes_k \mathcal{O}_C, W \otimes_k \mathcal{O}_C)$. Thus to prove Serre duality, we want to show that the map

$$\alpha: \text{Hom}(\mathcal{L}, \omega_C) \longrightarrow \text{Hom}(H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C, H^1(C, \omega_C) \otimes_k \mathcal{O}_C), \quad \phi \mapsto H^1(\phi) \otimes 1.$$

By naturality of the map δ , for a map $\phi: \mathcal{L} \rightarrow \omega_C$, we have a commutative square

$$\begin{array}{ccc} \mathcal{L} \otimes \omega_C^{-1} & \xrightarrow{\delta} & H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C \\ \phi \otimes 1 \downarrow & & \downarrow H^1(\phi) \otimes 1 \\ \omega_C \otimes \omega_C^{-1} & \xrightarrow{\sim} & H^1(C, \omega_C) \otimes_k \mathcal{O}_C. \end{array}$$

Since the bottom map is an isomorphism, we can treat the left map $\phi \otimes 1$ as the (pre)composition of the right map $H^1(\phi) \otimes 1$ with δ . We thus have a candidate for an inverse to α , namely

$$\beta: \text{Hom}(H^1(C, \mathcal{L}), H^1(C, \omega_C)) \longrightarrow \text{Hom}(\mathcal{L} \otimes \omega_C^{-1}, \omega_C \otimes \omega_C^{-1}) = \text{Hom}(\mathcal{L}, \omega_C), \quad \psi \mapsto ((\psi \otimes 1) \circ \delta) \otimes \omega_C.$$

We have just observed that the composition $\beta \circ \alpha$ is the identity, in particular α is injective.

To finish, it suffices to show β is injective. Suppose $\psi: H^1(C, \mathcal{L}) \rightarrow H^1(C, \omega_C)$ is such that $(\psi \otimes 1) \circ \delta = 0$. This means that $\delta_P \circ \psi = 0$ for every $P \in C$. Therefore it remains to show the following:

Lemma 18.5.5. *The space $H^1(C, \mathcal{L})$ is spanned by the images of δ_P for all $P \in C$.*

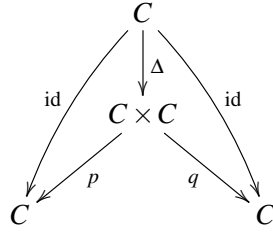
Proof. For distinct points P_1, \dots, P_r , Lemma 18.5.2 implies that the sum of the images of $\delta_{P_1}, \dots, \delta_{P_r}$ equals the image of the boundary map for (18.5.5), which is the kernel of the next map

$$H^1(C, \mathcal{L}) \longrightarrow H^1(C, \mathcal{L}(P_1 + \dots + P_r)).$$

It therefore suffices to find P_1, \dots, P_r for which the latter group is zero. By Corollary 18.5.3, this will be the case as long as $r > \deg(\omega_C) - \deg(\mathcal{L})$. \square

18.6. Construction of the map δ

In order to make sense of $\mathcal{L}(P)$ for a varying $P \in C$, we make use of the diagonal $\Delta \subseteq C \times C$, which is a divisor on the smooth projective surface $C \times C$. Let $p, q: C \times C \rightarrow C$ be the two projections



and consider the invertible sheaf $q^* \mathcal{L}$ and its twist $q^* \mathcal{L}(\Delta) = q^* \mathcal{L} \otimes \mathcal{O}_{C \times C}(\Delta)$. The restriction of the latter to $p^{-1}(P) = \{P\} \times C \simeq C$ is $\mathcal{L}(P)$. On $C \times C$, we have the short exact sequence

$$0 \longrightarrow q^* \mathcal{L} \longrightarrow q^* \mathcal{L}(\Delta) \longrightarrow q^* \mathcal{L}(\Delta)|_{\Delta} \longrightarrow 0.$$

Recall that (identifying the diagonal Δ with C via p) we have

$$\omega_C = \mathcal{I}_{\Delta} / \mathcal{I}_{\Delta}^2 = \mathcal{I}_{\Delta}|_{\Delta} = \mathcal{O}_{C \times C}(-\Delta)|_{\Delta}$$

and hence the rightmost sheaf $q^* \mathcal{L}(\Delta)|_{\Delta}$ is identified with $\Delta_*(\mathcal{L} \otimes \omega_C^{-1})$, which is the source of our postulated map δ .

In order to construct δ , we need to consider ‘‘cohomology in a family,’’ i.e. the derived functors $R^i p_*$ of the first projection $p: C \times C \rightarrow C$. They are defined in the same way as cohomology (say, using flabby resolutions), but replacing global sections with $p_*(-)$, and turn short exact sequences on the source $C \times C$ into long exact sequences of sheaves on the target C . In particular, we get the boundary map

$$\delta: \mathcal{L} \otimes \omega_C^{-1} = p_*(p^* \mathcal{L}(\Delta)|_{\Delta}) \longrightarrow R^1 p_* p^* \mathcal{L}.$$

As we shall see, the target is $H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$, and the fiber of δ at $P \in C$ is δ_P .

Proposition 18.6.1. *There is a functorial way of associating, to every invertible sheaf \mathcal{L} on C , a map of coherent sheaves*

$$\delta: \mathcal{L} \otimes \omega_C^{-1} \longrightarrow H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$$

such that for every $P \in C$ we have a commutative square

$$\begin{array}{ccc} (\mathcal{L} \otimes \omega_C^{-1})(P) & \xrightarrow{\delta(P)} & (H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C)(P) \\ \cong \downarrow & & \parallel \\ k & \xrightarrow{\delta_P} & H^1(C, \mathcal{L}). \end{array}$$

(In words: the fiber of δ at P is δ_P , up to a choice of basis of $(\mathcal{L} \otimes \omega_C^{-1})(P)$.)

Proof. As we have mentioned, this follows from the general machinery of higher direct images. Let us construct the map by hand. Let $C = U_0 \cup U_1$ be an affine open cover of C and let $U_{01} = U_0 \cap U_1$ (which is also affine). (As we have seen, any non-constant rational function on C gives a finite map $f: C \rightarrow \mathbb{P}^1$, and we can take $U_0 = C \setminus f^{-1}(\infty)$ and $U_1 = C \setminus f^{-1}(0)$.) As in the construction of the Čech complex, the inclusions $j_i: C \times U_i \hookrightarrow C \times C$ are affine, and for a coherent sheaf \mathcal{F} on $C \times C$, we have an exact sequence of quasi-coherent sheaves

missing formula

Applying p_* we obtain a complex of sheaves on C , exact on the left

$$0 \longrightarrow p_*\mathcal{F} \longrightarrow p_*\mathcal{C}^0(\mathcal{F}) \longrightarrow p_*\mathcal{C}^1(\mathcal{F}) \longrightarrow 0$$

Moreover, for any affine open $U \subseteq C$, the restriction of the above sequence to U is the (complex of sheaves associated to) the Čech complex of $\mathcal{F}|_{U \times C}$ and the open cover of $U \times C$ by $U \times U_i$.

Applying this to the terms of the short exact sequence

$$0 \longrightarrow q^*\mathcal{L} \longrightarrow q^*\mathcal{L}(\Delta) \longrightarrow \Delta_*(\mathcal{L} \otimes \omega_C^{-1}) \longrightarrow 0$$

we obtain an exact diagram of the form

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p_*q^*\mathcal{L} & \longrightarrow & p_*\mathcal{C}^0(q^*\mathcal{L}) & \xrightarrow{\beta} & p_*\mathcal{C}^1(q^*\mathcal{L}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p_*q^*\mathcal{L}(\Delta) & \longrightarrow & p_*\mathcal{C}^0(q^*\mathcal{L}(\Delta)) & \longrightarrow & p_*\mathcal{C}^1(q^*\mathcal{L}(\Delta)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & p_*\Delta_*(\mathcal{L} \otimes \omega_C^{-1}) & \longrightarrow & p_*\mathcal{C}^0(\Delta_*(\mathcal{L} \otimes \omega_C^{-1})) & \xrightarrow{\gamma} & p_*\mathcal{C}^1(\Delta_*(\mathcal{L} \otimes \omega_C^{-1})) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

(the two rightmost columns are exact since the compositions $p \circ j_i$ are affine). Snake lemma yields a map

$$\delta: \mathcal{L} \otimes \omega_C^{-1} = p_*\Delta_*(\mathcal{L} \otimes \omega_C^{-1}) \longrightarrow \text{coker}(\beta).$$

On any affine $U \subseteq C$, the top row is the Čech complex of \mathcal{L} tensored with $\mathcal{O}(U)$. Thus $\text{coker}(\beta) \simeq H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C$, and we obtain the desired map

$$\delta: \mathcal{L} \otimes \omega_C^{-1} = p_*\Delta_*(\mathcal{L} \otimes \omega_C^{-1}) \longrightarrow H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C.$$

Clearly, the construction is functorial in \mathcal{L} . (Technically speaking, we have not checked that it is also independent of the choice of the open cover.)

For a point $P \in C$, we make a further pull-back of this diagram to P . The columns remain exact since the stalks of modules on the bottom at P are torsion-free and hence flat. Moreover, we have $\text{coker}(\beta)(P) = \text{coker}(\beta(P))$. Finally, we need to argue that $\ker(\gamma)(P) = \ker(\gamma(P))$. Since the bottom row is simply the sheafy Čech complex of $\mathcal{L} \otimes \omega_C^{-1}$, the map γ is surjective. Since its target again is a flat \mathcal{O}_C -module, we can argue as for the left-exactness of the columns. The outcome of this analysis is that we have a commutative square

$$\begin{array}{ccc} (\mathcal{L} \otimes \omega_C^{-1})(P) & \xrightarrow{\delta(P)} & (H^1(C, \mathcal{L}) \otimes_k \mathcal{O}_C)(P) \\ \simeq \downarrow & & \downarrow \simeq \\ k & \xrightarrow{\delta_P} & H^1(C, \mathcal{L}) \end{array}$$

as desired. □

19. Lecture 19 (Apr 14): More projective tools

Our next big goal is to develop intersection theory on surfaces. For this, we need two more tools: the notion of an ample line bundle and the Bertini theorem on hyperplane sections. \mathcal{A}

Recommended reading: Hartshorne II 7 and V 1, Kempf §10.9.

19.1. Line bundles and maps to projective space

Let \mathcal{F} be a coherent sheaf on a projective variety X . Then the vector space $\Gamma(X, \mathcal{F})$ is finite-dimensional, and we can make it into a free \mathcal{O}_X -module $\Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_X$. We have a canonical map

$$\eta: \Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{F} \quad (19.1.1)$$

which on an open $U \subseteq X$ is the map

$$s \otimes f \mapsto f \cdot (s|_U) \quad : \quad \Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U).$$

(If we denote by p the unique map $X \rightarrow \star$ to the point, then this map is simply the counit $p^* p_* \mathcal{F} \rightarrow \mathcal{F}$.)

Definition 19.1.1. We say that a coherent sheaf \mathcal{F} on a projective variety X is **globally generated** if the map (19.1.1) is surjective.

We note right away that \mathcal{F} is globally generated if and only if it is a quotient of \mathcal{O}_X^n for some $n \geq 0$. In particular, if \mathcal{F} is globally generated and $\mathcal{F} \rightarrow \mathcal{F}'$ is surjective, then \mathcal{F}' is globally generated.

Examples 19.1.2. 1. A free sheaf \mathcal{O}_X^n is globally generated;

2. For every closed $Z \subseteq X$, the sheaf $\mathcal{O}_Z = \mathcal{O}_X / \mathcal{I}_Z$ is globally generated.

3. If \mathcal{F} and \mathcal{F}' are globally generated, then so is $\mathcal{F} \otimes \mathcal{F}'$. This follows from right-exactness of \otimes .

4. An invertible sheaf \mathcal{L} is globally generated if and only if for every $x \in X$ there exists an $s \in H^0(X, \mathcal{L})$ with $s(x) \neq 0$. A point $x \in X$ such that $s(x) = 0$ for all $s \in H^0(X, \mathcal{L})$ is called a base point, and we sometimes call a globally generated invertible sheaf **base-point free**.

5. For $\mathcal{L} = \mathcal{O}(1)$ on $X = \mathbb{P}^n$, the map (19.1.1) is

$$(x_0, \dots, x_n): \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \quad (19.1.2)$$

(as we have seen already, it is surjective, with kernel $\Omega_{\mathbb{P}^n}^1(1)$).

6. Serre's theorem (Lecture 16, Theorem 16.2.5 and Remark 16.2.6) is equivalent to the statement that for every coherent sheaf \mathcal{F} on \mathbb{P}^n the sheaf $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$ is globally generated for $d \gg 0$.

The following lemma describes the universal property of \mathbb{P}^n . In its statement we use the following terminology: for a coherent sheaf \mathcal{F} on a variety X , an **invertible sheaf quotient** of \mathcal{F} is a surjection $\mathcal{F} \rightarrow \mathcal{L}$ onto an invertible sheaf \mathcal{L} , where we deem $\phi: \mathcal{F} \rightarrow \mathcal{L}$ and $\phi': \mathcal{F} \rightarrow \mathcal{L}'$ equal if $\ker(\phi) = \ker(\phi')$ (or equivalently if $\mathcal{L} \simeq \mathcal{L}'$ under \mathcal{F}). With this convention, invertible sheaf quotients of \mathcal{F} form a set. Equation (19.1.2) exhibits a particular invertible sheaf quotient of $\mathcal{F} = \mathcal{O}_X^{n+1}$ on $X = \mathbb{P}^n$. For any map $f: Y \rightarrow X$, the right-exactness of f^* shows that f^* maps invertible sheaf quotients of \mathcal{F} to invertible sheaf quotients of $f^* \mathcal{F}$.

Lemma 19.1.3. For any algebraic set (or scheme) X , the map

$$\mathrm{Hom}(X, \mathbb{P}^n) \xrightarrow{\sim} \{\text{invertible sheaf quotients of } \mathcal{O}_X^{n+1}\}, \quad (f: X \rightarrow \mathbb{P}^n) \mapsto f^*((19.1.2)).$$

is bijective.

Before giving the proof, let us discuss the intuition behind this result. For this, it is better to formulate it in a coordinate-free manner. Let $V \simeq k^{n+1}$ be a k -vector space of dimension $n + 1$. Then

$$\mathbb{P}V = (V \setminus \{0\})/k^\times \simeq \mathbb{P}^n$$

parametrizes lines (one-dimensional linear subspaces) in V . We have $\Gamma(\mathbb{P}V, \mathcal{O}(1)) = V^\vee$, and $\mathbb{P}V^\vee$ parametrizes hyperplanes in V , or equivalently one-dimensional quotients of V . A correctly defined “family of one-dimensional quotients of V parametrized by X ” is precisely an invertible sheaf quotient of the free \mathcal{O}_X -module $V \otimes_k \mathcal{O}_X$, and thus such quotients should be in bijection with maps $X \rightarrow \mathbb{P}V^\vee$.

Proof. Let us describe the inverse map (details left to the reader). Let $s = (s_0, \dots, s_n): \mathcal{O}_X^{n+1} \rightarrow \mathcal{L}$ be a surjective map, where $s_i \in \Gamma(X, \mathcal{L})$. If $U_i = D(s_i) = \{x \in X : s_i(x) \neq 0\}$, then surjectivity of s means that $X = \bigcup U_i$. On U_i , the section s_i trivializes \mathcal{L} , meaning that $s_i|_{U_i}: \mathcal{O}_{U_i} \rightarrow \mathcal{L}|_{U_i}$ is an isomorphism. Accordingly, there exist unique $x_{j/i} \in \mathcal{O}_{U_i}$ such that $s_j = x_{j/i}s_i$ on U_i (we have $x_{i/i} = 1$). The elements $x_{j/i}$ for a fixed i induce a morphism $f_i: U_i \rightarrow D(x_i) \subseteq \mathbb{P}^n$. It is then straightforward to check that the maps f_i agree on the overlaps, producing a map $f: X \rightarrow \mathbb{P}^n$, that $\mathcal{L} \simeq f^*\mathcal{O}(1)$, and $s_i = f^*(x_i)$. \square

Let us reformulate what we have learned in terms of divisors. The upshot of the above lemma is that for a globally generated invertible sheaf \mathcal{L} on a smooth projective variety X we get a map

$$\phi_{\mathcal{L}}: X \longrightarrow \mathbb{P}V^\vee, \quad V = H^0(X, \mathcal{L}).$$

It maps a point $x \in X$ to the hyperplane in V defined as the kernel of the “evaluation at x ” map $H^0(X, \mathcal{L}) \rightarrow \mathcal{L}(x) \simeq k$ (which is a hyperplane precisely because \mathcal{L} is globally generated).

On the other hand, since invertible sheaves correspond to linear equivalence classes of divisors (divisors modulo principal ones) we have a bijection

$$\mathbb{P}V = \{D \geq 0 : \mathcal{L} \simeq \mathcal{O}_X(D)\}$$

with the set of effective divisors in the linear equivalence class \mathcal{L} . This bijection identifies a nonzero $s \in H^0(X, \mathcal{L})$ (up to scaling) with the effective divisor $D = V(s)$. Thus the map $\phi_{\mathcal{L}}$ maps a point $x \in X$ to the set of those effective divisors D linearly equivalent to \mathcal{L} which contain x in their support. (Compare this with the discussion at the beginning of [Hartshorne, II 6].)

19.2. Ample invertible sheaves

In algebraic geometry of the French tradition, one often turns theorems into definitions. We define an invertible sheaf to be *ample* if it satisfies the assertion of Serre’s theorem (Lecture 16, Theorem 16.2.5+Remark 16.2.6).

Definition 19.2.1. Let \mathcal{L} be an invertible sheaf on a projective variety X .

- (a) We say that \mathcal{L} is **ample** if for every coherent sheaf \mathcal{F} on X there exists an n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for $n \geq n_0$.
- (b) We say that \mathcal{L} is **very ample** if there exists a closed immersion $i: X \hookrightarrow \mathbb{P}^n$ for some $n \geq 0$ such that $\mathcal{L} \simeq i^*\mathcal{O}(1)$.

Remark 19.2.2. Now is as good time to review our notational conventions since they might be confusing.

- The direct sum of $n \geq 1$ copies of a coherent sheaf \mathcal{F} is denoted by \mathcal{F}^n . When we want to be more precise, we write $\mathcal{F}^{\oplus n}$ for the same thing.
- ...unless the coherent sheaf is an invertible sheaf \mathcal{L} , in which case \mathcal{L}^n typically means the tensor power $\mathcal{L}^{\otimes n}$ (defined for all $n \in \mathbb{Z}$ with the usual convention $\mathcal{L}^{-n} = (\mathcal{L}^\vee)^n$).

- ...unless the invertible sheaf is the trivial sheaf \mathcal{O}_X , in which case \mathcal{O}_X^n again means $\mathcal{O}_X^{\oplus n}$.
- For a divisor D and a coherent sheaf \mathcal{F} we denote by $\mathcal{F}(D)$ the tensor product $\mathcal{F} \otimes \mathcal{O}_X(D)$.
- Sometimes we denote a chosen ample line bundle by $\mathcal{O}(1)$ (without referencing an ambient projective space), and then we write $\mathcal{F}(n)$ for $\mathcal{F} \otimes \mathcal{O}(1)^n$.
- We say that a divisor D is globally generated or ample if the corresponding invertible sheaf $\mathcal{O}_X(D)$ has that property.

The following lemma gathers some basic facts about ample and very ample invertible sheaves.

Lemma 19.2.3. *Let \mathcal{L} be an invertible sheaf on a projective variety X .*

- (a) *If \mathcal{L} is very ample, then it is ample.*
- (b) *If \mathcal{L} is (very) ample, then \mathcal{L}^n is (very) ample for every $n \geq 1$.*
- (c) *If \mathcal{L}^n is ample for some $n \geq 1$, then \mathcal{L} is ample.*
- (d) *If $f: Y \rightarrow X$ is a finite morphism and \mathcal{L} is ample, then $f^*\mathcal{L}$ is ample.*
- (e) *If \mathcal{L} is ample, then for every invertible sheaf \mathcal{M} on X , the sheaf $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \gg 0$.*
- (f) *If \mathcal{L} is (very) ample and \mathcal{M} is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is (very) ample.*

Proof. (a) This is Serre's theorem.

(b) For "ample" this is obvious and for "very ample" it suffices to consider the case $X = \mathbb{P}^m$ and $\mathcal{L} = \mathcal{O}(1)$ (the "Veronese embedding").

(c) Suppose \mathcal{L}^n is ample and let \mathcal{F} be a coherent sheaf. Consider the sheaves $\mathcal{F} \otimes \mathcal{L}^i$ for $i = 0, \dots, n-1$. Then for $m \gg 0$ each $\mathcal{F} \otimes \mathcal{L}^i \otimes \mathcal{L}^{mn}$ will be globally generated, and the assertion follows by writing arbitrary integers as $mn + i$ with $i < n$.

(d) Let \mathcal{F} be a coherent sheaf on Y . Then $f_*\mathcal{F}$ is a coherent sheaf on X , and hence for $n \gg 0$ the sheaf $(f_*\mathcal{F}) \otimes \mathcal{L}^n$ is globally generated. On the other hand, we have an isomorphism ("projection formula")

$$(f_*\mathcal{F}) \otimes \mathcal{L}^n \simeq f_*(\mathcal{F} \otimes (f^*\mathcal{L})^n),$$

and we want $\mathcal{F} \otimes (f^*\mathcal{L})^n$ to be globally generated. To finish we observe (renaming the sheaf) that if $f_*\mathcal{F}$ is globally generated then so is \mathcal{F} . Indeed, suppose we have a surjection $\mathcal{O}_X^a \rightarrow f_*\mathcal{F}$, then by adjunction we obtain a map $\mathcal{O}_Y^a = f^*(\mathcal{O}_X^a) \rightarrow \mathcal{F}$, which is again surjective.

(e), (f) Omitted. □

Theorem 19.2.4 (Hartshorne II 7.6 + Exercise II 7.5(e)). *Let \mathcal{L} be an invertible sheaf on a projective variety X . If \mathcal{L} is ample, then \mathcal{L}^n is very ample for $n \gg 0$.*

Corollary 19.2.5. *If \mathcal{L} is ample, then for every coherent sheaf \mathcal{F} on X there exists an n_0 such that for all $n \geq n_0$ we have $H^q(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for $q > 0$.*

Proof. If \mathcal{L} is very ample, this follows from the case $X = \mathbb{P}^m$ and $\mathcal{L} = \mathcal{O}(1)$ handled previously (Lecture 17, 17.5.3). For the general case, suppose that \mathcal{L}^m is very ample by the previous theorem, then apply the very ample case to the sheaves $\mathcal{F} \otimes \mathcal{L}^i$ for $i = 0, \dots, m-1$. □

19.3. Hyperplane sections and Bertini

Let \mathcal{L} be a very ample invertible sheaf on X , let $s \in \Gamma(X, \mathcal{L})$ be a nonzero section, and let $Y = V(s)$ be its zero locus (considered as an effective divisor). If $i: X \hookrightarrow \mathbb{P}^n$ is a closed immersion for which $\mathcal{L} \simeq i^*\mathcal{O}(1)$, and if s is the image of an element (linear form) $\ell \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$, then $Y = X \cap H$ for a hyperplane $H = V(\ell) \subseteq \mathbb{P}^n$. This is why the zero sets of sections of (very) ample line bundles are called *hyperplane sections*.

Theorem 19.3.1 (Bertini). *Let $X \subseteq \mathbb{P}V$ (where $V \simeq k^{n+1}$) be a smooth projective variety of dimension d . Then there exists a dense open $U \subseteq \mathbb{P}V^\vee$ such that for every hyperplane $H \in U$ the intersection $H \cap X$ is smooth of dimension $d - 1$.*

Proof. The key point is that $H \cap X$ is a smooth divisor in a neighborhood of a point $x \in H \cap X$ if and only if H does not contain the tangent space $T_x X$. Consider the set of “bad pairs”

$$Z = \{(x, H) \in X \times \mathbb{P}V^\vee : T_x X \subseteq H\},$$

a closed subset of $X \times \mathbb{P}V^\vee$. Then the assertion holds for $H \in \mathbb{P}V^\vee$ if and only if $H \notin q(Z)$ where $q: Z \rightarrow \mathbb{P}V^\vee$ is the projection $(x, H) \mapsto H$. The subset $q(Z) \subseteq \mathbb{P}V^\vee$ is closed (since X is complete, so that q is a closed map), and the desired open U will be its complement. Thus, we must show that $q(Z)$ is not the whole of $\mathbb{P}V^\vee$.

We will prove that $q(Z) \neq \mathbb{P}V^\vee$ by showing that $\dim(Z) < n = \dim \mathbb{P}V^\vee$. To this end, consider the other projection $p: Z \rightarrow X$ given by $(x, H) \mapsto x$. The fiber of p above $x \in X$ is the set of hyperplanes in $\mathbb{P}V \simeq \mathbb{P}^n$ containing the d -dimensional linear subspace $T_x X$. It can be identified with the set of hyperplanes in the quotient space $V/C(T_x X)$ where $C(T_x X)$ is the cone over $T_x X$, a linear subspace of V of dimension $d + 1$. Thus the fiber $p^{-1}(x)$ can be identified with $\mathbb{P}(V/C(T_x X))^\vee \simeq \mathbb{P}^{n-d-1}$, which has dimension $n - d - 1$. Consequently, we have

$$\dim(Z) = \dim(X) + \dim(\text{general fiber of } p) = d + (n - d - 1) = n - 1 < n. \quad \square$$

Corollary 19.3.2. *Let \mathcal{L} be an invertible sheaf on a smooth projective variety X . Then there exist smooth prime divisors $D_1, \dots, D_r, E_1, \dots, E_s$ on X such that $D_i \cap D_j = \emptyset = E_i \cap E_j$ for $i \neq j$ such that*

$$\mathcal{L} \simeq \mathcal{O}_X((D_1 + \dots + D_r) - (E_1 + \dots + E_s)).$$

Remark 19.3.3. We can choose the D_1, \dots, D_r and E_1, \dots, E_s to be ample and if $\dim(X) \geq 2$ we will have $r = s = 1$.

Proof. Let \mathcal{M} be a very ample invertible sheaf. Choose $n > 0$ such that both \mathcal{M}^n and $\mathcal{M}^n \otimes \mathcal{L}$ are very ample (cf. Lemma 19.2.3(e)). By Bertini (Theorem 19.3.1) we can find $s \in H^0(X, \mathcal{M}^n \otimes \mathcal{L})$ and $t \in H^0(X, \mathcal{M}^n)$ such that $D = V(s)$ and $E = V(t)$ are smooth. Then $\mathcal{M}^n \otimes \mathcal{L} \simeq \mathcal{O}_X(D)$ and $\mathcal{M}^n \simeq \mathcal{O}_X(E)$, so that $\mathcal{L} \simeq \mathcal{O}_X(D - E)$. \square

19.4. Intersections on surfaces

Let X be a smooth projective surface and let C and D be two divisors on X . Our goal is to associate to this pair an integer $(C.D)$ called the **intersection number**. Our basic requirement is that if C and D are distinct prime divisors (so that $C \cap D$ is a finite set), then $(C.D)$ is the number of points in the intersection, counted with appropriate multiplicity:

$$(C.D) = \sum_{x \in C \cap D} i(C, D; x), \quad i(C, D; x) = \dim_k \mathcal{O}_{X,x} / (\mathcal{J}_{C,x} + \mathcal{J}_{D,x}).$$

We also want $(C.D)$ to be bilinear and symmetric, and to depend only on the linear equivalence classes of C and D .

Suppose that C and D are smooth prime divisors intersecting transversally (meaning that their tangent lines $T_x C, T_x D \subseteq T_x X$ are distinct), meaning that $i(C, D; x) = 1$ for every $x \in C \cap D$. (We make this assumption in order for $C \cap D$ to be reduced, so that we can avoid talking about schemes.) The trick to extending (C, D) to arbitrary divisors is to regard $\#(C \cap D)$ as the Euler characteristic $\chi(\mathcal{O}_{C \cap D})$ (the higher cohomology groups of $\mathcal{O}_{C \cap D}$ are zero). In this case we have an exact sequence (a special case of a “Koszul complex”)⁹

$$0 \longrightarrow \mathcal{O}_X(-C-D) \longrightarrow \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0$$

(recall that $\mathcal{J}_C = \mathcal{O}_X(-C)$ and $\mathcal{J}_D = \mathcal{O}_X(-D)$), from which we infer that

$$(C, D) = \chi(\mathcal{O}_{C \cap D}) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C-D)).$$

The same argument (now with multiplicities taken into account) applies to an arbitrary pair of curves C and D without common components (this ensures the exactness of the Koszul complex on the left). This motivates the following definition (cf. Kempf §10.9, Hartshorne V Ex. 1.1):

Definition 19.4.1. Let C and D be divisors on a smooth projective surface X . Their **intersection number** is defined as

$$(C, D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C-D)).$$

Clearly, the intersection number (C, D) depends only on the linear equivalence classes of C and D and is symmetric. Its being bilinear follows from the lemma below.

Lemma 19.4.2. Suppose that $D = D_1 + \cdots + D_r - E_1 - \cdots - E_s$ for smooth curves $D_1, \dots, D_r, E_1, \dots, E_s$ as in Corollary 19.3.2. Then for every divisor C we have

$$(C, D) = \sum \deg(\mathcal{O}_X(C)|_{D_i}) - \sum \deg(\mathcal{O}_X(C)|_{E_i}).$$

In particular, if D is a smooth curve, we have $(C, D) = \deg(\mathcal{O}_X(C)|_D)$.

Proof. The proof follows a straightforward calculation using various restriction exact sequences — see [Kempf, Lemma 10.9.2]. \square

Example 19.4.3 (Bézout theorem). On $X = \mathbb{P}^2$, we have an isomorphism $\deg: \text{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$ sending $\mathcal{O}(1)$ to 1. Then $(C, D) = \deg(C) \cdot \deg(D)$. This implies that two irreducible curves $C, D \subseteq X$ without common components have exactly $\deg(C) \cdot \deg(D)$ points in common, counted with multiplicity.

Example 19.4.4 (Exceptional divisor). Let $\pi: X = \text{Bl}_P \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the blowup of a point P in \mathbb{P}^2 . The fiber $E = \pi^{-1}(P)$ is isomorphic to \mathbb{P}^1 . We shall prove that $(E, E) = -1$. Since this is negative, we conclude from this that there is no effective $E' \sim E$ with $E' \neq E$ (otherwise $(E, E) = (E, E')$ would count the points in $E \cap E'$ and hence be non-negative).

Let $L \subseteq \mathbb{P}^2$ be a line through P and let $L' \subseteq \mathbb{P}^2$ be a map not through P . Let $\tilde{L} \subseteq X$ and $\tilde{L}' \subseteq X$ be their strict transforms. Then $\pi^{-1}(L) = \tilde{L} + E$ and $\pi^{-1}(L') = \tilde{L}'$. Since $\mathcal{O}_{\mathbb{P}^2}(L) = \mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^2}(L')$, we have

$$\mathcal{O}_X(\tilde{L} + E) \simeq \pi^* \mathcal{O}_{\mathbb{P}^2}(L) \simeq \pi^* \mathcal{O}_{\mathbb{P}^2}(L') \simeq \mathcal{O}_X(\tilde{L}'),$$

i.e. $\tilde{L} + E \sim \tilde{L}'$. Thus

$$\underbrace{(\tilde{L}', \tilde{L}')}_{=1} = \underbrace{(\tilde{L}, \tilde{L})}_{=0} + 2 \underbrace{(\tilde{L}, E)}_{=1} + (E, E),$$

and hence $(E, E) = -1$.

⁹See the last subsection.

Example 19.4.5 (The diagonal). Let C be a curve of genus g , let $X = C \times C$, and let $\Delta: C \hookrightarrow X$ be the diagonal. Then $\mathcal{O}(-\Delta) \simeq \omega_C$. It follows that

$$(\Delta, \Delta) = \deg(\mathcal{O}_X(\Delta)|_\Delta) = \deg(\omega_C^{-1}) = 2 - 2g.$$

Example 19.4.6. Let $f: Y \rightarrow X$ be a finite flat morphism of surfaces of degree d and let C and D be divisors on X . Then

$$(f^*C \cdot f^*D) = d \cdot (C \cdot D).$$

Lemma 19.4.7. *If $\mathcal{L} = \mathcal{O}_X(C)$ is ample and D is effective then $(C \cdot D) \geq 0$, with equality if and only if $D = 0$. Moreover, we have $(C \cdot C) > 0$.*

Proof. If C is ample then for every smooth curve $D \subseteq X$ the restriction $\mathcal{L}|_D$ is ample, and hence $(C \cdot D) = \deg(\mathcal{L}|_D) > 0$. For the general case, replacing C with nC for $n \gg 0$ we may assume that C is very ample, in which case we can find a smooth curve $C' \sim C$ which intersects D transversally. See [Hartshorne, V Lemma 1.3] for the details. \square

The following theorem gives a converse to the above lemma.

Theorem 19.4.8 (Nakai–Moishezon criterion for ampleness, Hartshorne V Theorem 1.10). *For a divisor C on X , the invertible sheaf $\mathcal{O}_X(C)$ is ample if and only if $(D \cdot D) > 0$ and $(C \cdot D) > 0$ for every irreducible curve $C \subseteq X$.*

The following theorem will be proved next week. It states that the intersection form on $\text{Pic}(X)$ has “signature $(1, n - 1)$.” We cannot formulate it like that since $\text{Pic}(X)$ is not a finite-dimensional vector space (though it can be done, see next week). By Lemma 19.4.7 above, we have $(H \cdot H) > 0$ if $\mathcal{O}_X(H)$ is ample, thus having “signature $(1, n - 1)$ ” should mean that the quadratic form $(D \cdot D)$ is negative-definite on the orthogonal to H :

Theorem 19.4.9 (Hodge index theorem). *If H is a divisor such that $\mathcal{O}_X(H)$ is ample and D a divisor such that $(D \cdot H) = 0$, then $(D \cdot D) \leq 0$, with equality if and only if $(D \cdot C) = 0$ for every divisor C .*

19.5. Problem session (Apr 14)

During the problem session we proved assertions (e) and (f) in Lemma 19.2.3. We then went through the material in §19.4 (the lectured covered only the first three subsections), proving the exactness of the Koszul complex and deducing the formula for $(C \cdot D)$ in terms of χ . We then proved Lemma 19.4.2 and deduced the formula in Example 19.4.5.

19.6. Bonus: exactness of the Koszul complex

The following result was used implicitly at the beginning of §19.4.

Definition 19.6.1. Let A be a local ring and let $x_1, \dots, x_n \in \mathfrak{m}_A$ be elements in its maximal ideal. We say that (x_1, \dots, x_n) form a **regular sequence** if for every $i = 1, \dots, n$, the image of x_i in $A/(x_1, \dots, x_{i-1})$ is a nonzerodivisor. If A is Noetherian, we say that A satisfies **Serre’s condition** (S_d) if it admits a regular sequence of length $n = \inf\{\dim(A), d\}$. We say that A is **Cohen–Macaulay** if it is Noetherian and admits a regular sequence of length $n = \dim(A)$.

Let A be any ring and let $x_1, \dots, x_n \in A$. For $0 \leq m \leq n$, let $\wedge^m A^n$ denote the free A -module of rank $\binom{n}{m}$ with basis $e_I = e_{i_1} \wedge \dots \wedge e_{i_m}$ where $I = \{i_1, \dots, i_m\}$, $1 \leq i_1 < \dots < i_m \leq n$. We define the differential $d: \wedge^m A^n \rightarrow \wedge^{m-1} A^n$ by the formula

$$d(e_I) = \sum_{i \in I} (-1)^j x_i \cdot e_{I \setminus \{i\}}.$$

For example, for $m = 1$, $d: A^n \rightarrow A$ is the map (x_1, \dots, x_n) . The **Koszul complex** is the resulting complex

$$K^\bullet(A; x_1, \dots, x_n) = \left[\wedge^n A^n \longrightarrow \wedge^{n-1} A^n \longrightarrow \dots \longrightarrow \wedge^2 A^n \longrightarrow A^n \longrightarrow A \right],$$

placed in degrees $[-n, 0]$. The Koszul complex is independent of the order of x_1, \dots, x_n , up to changing signs in the differentials. The projection $A \rightarrow A/(x_1, \dots, x_n)$ gives a map $K^\bullet(A; x_1, \dots, x_n) \rightarrow A/(x_1, \dots, x_n)$ (the target placed in degree zero).

Proposition 19.6.2. *Let A be a Noetherian local ring and $x_1, \dots, x_n \in \mathfrak{m}_A$. The following are equivalent:*

- (a) (x_1, \dots, x_n) is a regular sequence;
- (b) $K^\bullet(A; x_1, \dots, x_n) \rightarrow A/(x_1, \dots, x_n)$ is a quasi-isomorphism (equivalently, $H^i(K^\bullet(A; x_1, \dots, x_n)) = 0$ for $i \neq 0$).

Proof. For $n = 1$, the Koszul complex is

$$K^\bullet(A, x_1) = [A \xrightarrow{x_1} A],$$

and the equivalence of (a) and (b) is clear. For the general case, note first that (x_1, \dots, x_n) is a regular sequence if and only if x_1 is a nonzerodivisor and (x_2, \dots, x_n) gives a regular sequence on A/x_1 . We shall complete the argument for (a) \Rightarrow (b) for $n = 2$, which is sufficient for §19.4 (see [Eisenbud, §17] for the general case). Tensoring the short exact sequences

$$0 \longrightarrow A \xrightarrow{x_i} A \longrightarrow A/x_i \longrightarrow 0$$

together, we obtain a diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cdots \\
 0 & \longrightarrow & A & \xrightarrow{x_1} & A & \longrightarrow & A/x_1 \longrightarrow 0 \\
 & & \downarrow x_2 & & \downarrow x_2 & & \downarrow x_2 \alpha \\
 0 & \longrightarrow & A & \xrightarrow{x_1} & A & \longrightarrow & A/x_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \cdots \longrightarrow & A/x_2 & \xrightarrow[\beta]{x_1} & A/x_2 & \longrightarrow & A/(x_1, x_2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Snake lemma gives $\ker(\alpha) \xrightarrow{\sim} \ker(\beta)$, i.e. x_1 is a nonzerodivisor in A/x_2 if and only if x_2 is a nonzerodivisor in A/x_1 . Thus if (x_1, x_2) is a regular sequence, then the diagram is exact also with the dotted arrows. A diagram chase then shows that

$$K^\bullet(A; x_1, x_2) = \left[A \xrightarrow{(x_1, -x_2)^T} A^2 \xrightarrow{(x_1, x_2)} A \right]$$

is exact in negative degrees, so that (b) holds. □

20. Lecture 20 (Apr 16): Surfaces

In the final lecture of the “core” part of the course, we further develop intersection theory on surfaces: Serre duality, Riemann–Roch, and the Hodge index theorem.

Recommended reading: Hartshorne V 1, Kempf §10

20.1. General Serre duality

Theorem 20.1.1. *Let X be a smooth projective variety of dimension d . For every locally free sheaf \mathcal{E} on X and every $0 \leq q \leq d$ there exists a canonical isomorphism*

$$H^q(X, \mathcal{E}) \simeq H^{n-q}(X, \mathcal{E}^\vee \otimes \omega_X)^\vee.$$

Remark 20.1.2. We shall only need the following corollary: if \mathcal{L} is an invertible sheaf on a smooth projective surface X , then

$$\dim H^2(X, \mathcal{L}) = \dim H^0(X, \mathcal{L}^{-1} \otimes \omega_X).$$

I do not know if there is a simple proof of this.

The following application of coherent cohomology was promised earlier in Lecture 15, Example 15.1.1.

Corollary 20.1.3. *Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety of dimension ≥ 2 and let $H \subseteq \mathbb{P}^n$ be a hyperplane. Then the intersection $H \cap X$ is connected.*

Proof. As explained in Lecture 15, Example 15.1.1, the result will follow once we know that $H^1(X, \mathcal{O}_X(-n)) = 0$ for $n \gg 0$. By Serre duality, this space is dual to $H^{\dim(X)-1}(X, \mathcal{O}_X(n) \otimes \omega_X)$. Since $\dim(X) - 1 > 0$, this last group will vanish for $n \gg 0$ (see Lecture 19, Lemma 19.2.2(d)). \square

20.2. Riemann–Roch for surfaces

Let X be a smooth projective surface. As in the case of curves, we fix a divisor K for which $\mathcal{O}_X(K) \simeq \omega_X$ and call it a/the canonical divisor.

Last time, we introduced the intersection numbers $(C.D)$ between divisors on X . Let us recall their properties in form of a theorem.

Theorem 20.2.1. *Let X be a smooth projective surface. There exists a unique symmetric bilinear pairing*

$$(-, -): \text{Div}(X) \times \text{Div}(X) \longrightarrow \mathbb{Z}$$

satisfying the following two properties:

- i. if $C \sim C'$ and $D \sim D'$ (linear equivalence), then $(C.D) = (C'.D')$;*
- ii. if C and D are distinct prime divisors (irreducible curves), then*

$$(C.D) = \dim_k H^0(X, \mathcal{O}_{C \cap D}) = \sum_{x \in C \cap D} i(C, D; x), \quad i(C, D; x) = \dim_k \mathcal{O}_{C \cap D, x},$$

where $\mathcal{O}_{C \cap D} = \mathcal{O}_X / \mathcal{J}_{C \cap D}$ for $\mathcal{J}_{C \cap D} = \mathcal{J}_C + \mathcal{J}_D$ (since we are not taking the radical, we are counting the intersection points with the correct multiplicity).

In addition, the intersection numbers $(C.D)$ enjoy the following properties:

- (a) For any divisors C and D , we have*

$$(C.D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-C-D)).$$

(b) If C is a smooth prime divisor, then for every divisor D we have

$$(C.D) = \deg(\mathcal{O}_X(D)|_C).$$

The Riemann–Roch theorem has a version for surfaces.

Theorem 20.2.2 (Riemann–Roch for surfaces). *Let X be a smooth projective surface and let D be a divisor on X . Then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D \cdot (D - K)) + \chi(\mathcal{O}_X).$$

Proof. Let us try to follow the strategy of proof of the Riemann–Roch theorem for curves. The statement holds trivially for $D = 0$, and it depends only on the linear equivalence class of D . Applying Bertini (Lecture 19, Corollary 19.3.2) inductively, we see that it suffices to show that if $C \subseteq X$ is a smooth curve, then the statement holds for D if and only if it holds for $D + C$, or equivalently that

$$\chi(\mathcal{O}_X(D + C)) - \chi(\mathcal{O}_X(D)) = \left(\frac{1}{2}((D + C) \cdot (D + C - K)) + \chi(\mathcal{O}_X) \right) - \left(\frac{1}{2}(D \cdot (D - K)) + \chi(\mathcal{O}_X) \right),$$

where the right-hand side simplifies further as $(C.D) - \frac{1}{2}(C.K) + \frac{1}{2}(C.C)$. Now the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + C) \longrightarrow \mathcal{O}_X(D + C)|_C \longrightarrow 0$$

gives

$$\chi(\mathcal{O}_X(D + C)) - \chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D + C)|_C),$$

which by Riemann–Roch for curves equals $\deg(\mathcal{O}_X(D + C)|_C) + 1 - g(C)$. By Lecture 19, Lemma 19.4.2, we have

$$\deg(\mathcal{O}_X(D + C)|_C) = ((D + C).C) = (D.C) + (C.C).$$

Moreover, by the adjunction formula (Lecture 17, Example 17.3.4) we have

$$2g(C) - 2 = \deg(\omega_C) = \deg(\omega_X(C)|_C) = (K.C) + (C.C),$$

and so $1 - g(C) = -\frac{1}{2}((K.C) + (C.C))$. Combining what we know yields the result. \square

As in the case of curves, we define $\ell(D) = \dim H^0(X, \mathcal{O}_X(D))$. Riemann–Roch combined with Serre duality (in the form of Remark 20.1.2) gives the following inequality.

Corollary 20.2.3. *For any divisor D we have*

$$\ell(D) + \ell(K - D) \geq (D \cdot (D - K)) + \chi(\mathcal{O}_X).$$

In particular, if $(D.D) > 0$ then for $n \gg 0$, either nD or $K - nD$ is effective.

Proof. For the first statement note that

$$\ell(D) + \ell(K - D) = \dim H^0(X, \mathcal{O}_X(D)) + \dim H^2(X, \mathcal{O}_X(D)) \geq \chi(\mathcal{O}_X(D)) = (D \cdot (D - K)) + \chi(\mathcal{O}_X).$$

For the second, note that the right-hand side of the inequality for the divisor nD equals

$$n^2(D.D) - n(D.K) + \chi(\mathcal{O}_X)$$

which will be positive for $n \gg 0$. \square

20.3. The Hodge index theorem

In the last lecture, we introduced the notion of an ample invertible sheaf. We call a divisor ample if the corresponding line bundle is ample. For the theorem below (along with its proof) it is important to remember the following:

- (1) If $X \subseteq \mathbb{P}^n$ then the restriction of $\mathcal{O}(1)$ is ample. In other words, if $H \subseteq \mathbb{P}^n$ is a hyperplane which does not contain X , then $X \cap H$ is an ample divisor.
- (2) If H is an ample divisor and C is an effective divisor, then $(H.C) > 0$. Moreover, the divisor nH is effective for $n \gg 0$, and consequently we have $(H.H) > 0$.
- (3) If H is an ample divisor and C is any divisor, then for $n \gg 0$ the divisor $C + nH$ is ample, effective, and has $H^q(X, \mathcal{O}_X(C + nH)) = 0$ for $q > 0$.

Theorem 20.3.1 (Hodge index theorem). *Let X be a smooth projective surface and let H be an ample divisor on X . Let D be another divisor, and suppose that $D \cdot H = 0$ and $D \cdot C \neq 0$ for some divisor C . Then $D^2 < 0$.*

Proof. Suppose first that $D^2 > 0$. Let $H' = D + mH$ for $m \gg 0$ such that H' is ample (Lecture 19, Lemma 19.2.2(f)). Then $H' \cdot D = D^2 + mD \cdot H > 0$. Therefore for $n \gg 0$ we have

$$H' \cdot (K - nD) = H' \cdot K - nD^2 < 0,$$

and hence $h^0(K - nD) = 0$. Therefore $h^2(nD) = 0$ for $n \gg 0$ by Serre duality.

The Riemann–Roch formula gives

$$h^0(nD) - h^1(nD) + h^2(nD) = \chi(\mathcal{O}_X(nD)) = nD(nD + K) + \chi(\mathcal{O}_X) = D^2 n^2 + O(n)$$

which is positive for $n \gg 0$. But since $h^2(nD) = 0$ for $n \gg 0$, we must have $h^0(nD) > 0$ for $n \gg 0$. But then $nD \cdot H > 0$ (being the intersection of an ample divisor with a non-trivial effective divisor), contradicting our assumption that $D \cdot H = 0$.

Suppose now that $D^2 = 0$. Let $C' = (H^2)C - (C \cdot H)H$, then $C' \cdot D = C \cdot D \neq 0$ but now also $C' \cdot H = 0$. Consider the family of divisors $nD + C'$. We have

$$(nD + C')^2 = n^2 D^2 + n(D \cdot C') + (C')^2 = n(D \cdot C') + (C')^2$$

which is a non-constant linear function. Therefore there exists an n such that $(nD + C')^2 > 0$. We can now apply the case $D^2 > 0$ to the divisor $nD + C'$ in place of D to get a contradiction. \square

Definition 20.3.2 (Néron–Severi group). Let X be a smooth projective surface. A divisor $D \subseteq X$ is **numerically trivial** if $D \cdot C = 0$ for every divisor C . We denote by $\text{NS}(X)$ the quotient of $\text{Pic}(X)$ by the subgroup consisting of numerically trivial divisors. We call it the **Néron–Severi group** of X . The intersection product induces a non-degenerate symmetric bilinear pairing on $\text{NS}(X)$.

Theorem 20.3.3 (Theorem of the base, Néron). *The group $\text{NS}(X)$ is a free group of finite rank.*

Definition 20.3.4. The rank of $\text{NS}(X)$ is called the **Picard rank** of X and is denoted by $\rho(X)$.

Corollary 20.3.5 (Equivalent form of the Hodge index theorem). *Equip the real vector space $\text{NS}(X)_{\mathbb{R}}$ with the intersection pairing. It has signature $(1, \rho(X) - 1)$.*

Remark 20.3.6 (Cones). Various kinds of line bundles give rise to convex cones in the real vector space $\text{NS}(X)_{\mathbb{R}}$. The **effective cone** $\text{Eff}(X)$ is spanned by the classes of effective divisors. Property (2) of ample divisors implies that the **ample cone** is the interior of the dual cone $\text{Eff}(X)^{\vee} = \text{Nef}(X)$ called the **nef cone** (“nef” stands for “numerically effective”). Property (3) states that the ample cone is contained in the effective cone and it has non-empty interior, so that the ray $C + \mathbb{R}_{>0} \cdot H$ for ample H eventually intersects the ample cone. The structure of the nef cone is important in higher-dimensional birational geometry.

We shall later need the following less straightforward corollary of the Hodge index theorem.

Proposition 20.3.7. *Let C be a smooth projective curve, let $P \in C$, and let $X = C \times C$. Let $C_1 = \{P\} \times C$ and $C_2 = C \times \{P\}$. Then for any divisors D_1, D_2 on X we have*

$$|(C_1 \cdot D_1)(C_2 \cdot D_2) + (C_1 \cdot D_2)(C_2 \cdot D_1) - (D_1 \cdot D_2)| \leq \sqrt{2(C_1 \cdot D_1)(C_2 \cdot D_1) - (D_1 \cdot D_1)} \\ \cdot \sqrt{2(C_1 \cdot D_2)(C_2 \cdot D_2) - (D_2 \cdot D_2)}.$$

Proof. We have $(C_1 \cdot C_2) = 1$ since the curves intersect transversally at the unique point (P, P) . Next, we check that $(C_1 \cdot C_1) = 0$; indeed, this intersection number equals the degree of $\mathcal{O}(C_1)|_{C_1}$, but the sheaf $\mathcal{O}_X(C_1)|_{C_1}$ is free (spanned by the image of a uniformizer of $\mathcal{O}_{C,P}$) so its degree is zero. Similarly $(C_2 \cdot C_2) = 0$. We are now in position to apply Exercise 6 on Problem Set 5. \square

21. Lecture 21 (Apr 21): Point counting over finite fields

In the final part of the course, we shall apply the theory we have developed to an arithmetic problem: the Riemann hypothesis for curves over finite fields.

Recommended reading:

- Hartshorne [Har77, Appendix C];
- notes by Sam Raskin¹⁰;
- the book [Lor96] *An Invitation to Arithmetic Geometry* by Dino Lorenzini (Chapters VIII and X).

The Weil conjectures for curves can also be deduced from the inequalities for correspondences in Kempf [Kem93, §10.8] (his proof does not require intersection theory on surfaces).

21.1. Varieties over non-closed fields and rational points

The formalism of algebraic varieties developed in this course is not so well equipped to deal with geometry over fields which are not algebraically closed, since a variety over a field might not have any points with coordinates that field (e.g. $V(x^2 + y^2 + 1)$ over \mathbb{R}). This is one more reason to use schemes, which work well over arbitrary base fields (or even over rings such as \mathbb{Z}). Instead of doing that, we circumvent the difficulties by fixing an embedding into an affine or projective space; this is a bit cumbersome, but will do for our purposes.

Definition 21.1.1. Let k be an algebraically closed field and let $k_0 \subseteq k$ be a subfield. We say that a closed subset X of \mathbb{A}^n is **defined over** k_0 if there exist polynomials $f_1, \dots, f_r \in k_0[T_1, \dots, T_n]$ such that $X = V(f_1, \dots, f_r)$. A point $x \in X$ is **rational over** k_0 if it is defined over k_0 , or equivalently if $x = (x_1, \dots, x_n)$ where $x_1, \dots, x_n \in k_0$. We write $X(k_0)$ for the set of all $x \in X$ which are rational over k_0 . We make the same definitions for $X \subseteq \mathbb{P}^n$ (in which case $f_1, \dots, f_r \in k_0[T_0, \dots, T_n]$ are supposed to be homogeneous).

Remark 21.1.2. 1. Obviously, being defined over k_0 depends on the embedding (already if X is a point). To see how to get a reasonable “hands-on” theory of varieties over k_0 using algebraic varieties over k , see [Mum99, §II.4].

2. This definition has many variants. For example, we could extend it to locally closed (or constructible) subsets. If X is a variety, we can define “divisors on X defined over k_0 .” A map $f: Y \rightarrow X$ between two varieties defined over k_0 (with respect to a pair of embeddings a projective spaces) is defined over k_0 if its graph (embedded using the Segre embedding) is defined over k_0 , and so on.

21.2. The Hasse–Weil zeta function

The Riemann zeta function $\zeta(s)$ of the complex variable s (with $\operatorname{Re}(s) > 1$) is

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p \frac{1}{1 - p^{-s}}.$$

This equality (the “Euler product”) is an analytic avatar of unique factorization of positive integers as products of powers of primes. It extends to a meromorphic function on the complex plane with a single pole at zero, and the **Riemann hypothesis** is the conjecture that the zeros of $\zeta(s)$ in the “critical strip” $0 \leq \operatorname{Re}(s) \leq 1$ lie on the line $\operatorname{Re}(s) = 1/2$. This statement has many equivalent forms, often stated in terms of the asymptotics of the prime counting function $\pi(x)$.

¹⁰<https://math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Raskin.pdf>

We now build an analog of the Riemann zeta function associated to a variety defined over a finite field. In fact, this is more than an analogy, as Remark 21.3.5 will explain why both are special cases of a more general construction. For this, we need to recall a few **facts about finite fields**. Let \mathbb{F}_q be a finite field of cardinality $q = p^c$ and let $\overline{\mathbb{F}}_q = \overline{\mathbb{F}}_p$ be a fixed algebraic closure. We can identify \mathbb{F}_q as the set of solutions of $T^q - T = 0$ inside $\overline{\mathbb{F}}_q$. More generally, for every $r \geq 1$, we the set \mathbb{F}_{q^r} of solutions of $T^{q^r} - T = 0$ is the unique extension of \mathbb{F}_q of degree r (and hence a field of cardinality q^r). The Galois group $\text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ is cyclic of order r , generated by the Frobenius automorphism $x \mapsto x^q$.

From now on we work in our usual algebraic geometry framework over the algebraically closed field $k = \overline{\mathbb{F}}_q$, so for example \mathbb{P}^n denotes the projective n -space over $k = \overline{\mathbb{F}}_q$. We use the terminology introduced in §21.1: a locally closed $X \subseteq \mathbb{P}^n$ is defined over \mathbb{F}_q if it is defined using homogeneous polynomials with coefficients in \mathbb{F}_q , and a point $x \in X$ is rational over \mathbb{F}_q if $x = (x_0 : \cdots : x_n)$ with $x_i \in \mathbb{F}_q$, or equivalently if $\{x\}$ is defined over \mathbb{F}_q . We write

$$X(\mathbb{F}_q) = \{x \in X : x \text{ is defined over } \mathbb{F}_q\}.$$

Since \mathbb{F}_q is finite, this is a finite set. Obviously if X is defined over \mathbb{F}_q , it is also defined over \mathbb{F}_{q^r} for all $r \geq 1$, and so the notation $X(\mathbb{F}_{q^r})$ makes sense. Since every point is defined over *some* finite extension of \mathbb{F}_q , we have

$$X = \bigcup_{r \geq 1} X(\mathbb{F}_{q^r}).$$

The basic questions we aim to answer are:

How big is the set $X(\mathbb{F}_q)$? How fast does the size of $X(\mathbb{F}_{q^r})$ grow with r ?

It is convenient to pack the numbers $\#X(\mathbb{F}_{q^r})$ into a kind of generating function.

Definition 21.2.1. Let $X \subseteq \mathbb{P}^n$ be a locally closed subset defined over \mathbb{F}_q . The **Hasse–Weil zeta function** of X over \mathbb{F}_q is the formal power series

$$Z(X/\mathbb{F}_q, T) = \exp\left(\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{q^r})}{r} T^r\right) \in \mathbb{Q}[[T]].$$

A nice property of the zeta function is that it can be computed “piece by piece.”

Remark 21.2.2 (Scissor relations). If X is the disjoint union of locally closed subsets X_1 and X_2 , then $\#X(\mathbb{F}_{q^r}) = \#X_1(\mathbb{F}_{q^r}) + \#X_2(\mathbb{F}_{q^r})$ and consequently

$$Z(X/\mathbb{F}_q, T) = Z(X_1/\mathbb{F}_q, T) \cdot Z(X_2/\mathbb{F}_q, T).$$

In the example below, and many times afterwards, we shall use the power series expansion

$$\log\left(\frac{1}{1 - \alpha T}\right) = \sum_{r \geq 1} \frac{\alpha^r}{r} T^r. \quad (21.2.1)$$

Example 21.2.3 (Projective space). Let $X = \mathbb{P}^n$, then we can write

$$X = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0,$$

and using the scissor relations we get

$$\#X(\mathbb{F}_{q^r}) = 1 + q^r + \cdots + q^{rn}.$$

Using (21.2.1) we turn this to

$$Z(\mathbb{P}^n/\mathbb{F}_q, T) = \frac{1}{(1 - T)(1 - qT) \cdots (1 - q^n T)}.$$

Example 21.2.4 (Fermat curves). For $d \geq 1$ prime to p , consider the Fermat curve

$$X_d = \{x^d + y^d + z^d = 0\} \subseteq \mathbb{P}^2.$$

Let $q = p^c$ be such that $q \equiv 1 \pmod{d}$. Obviously X_d is defined over \mathbb{F}_p and hence over \mathbb{F}_q . According to the rather ingenious computation in [IR90, §11.3], we have

$$Z(X_d/\mathbb{F}_q, T) = \frac{\prod_{\alpha\beta\gamma=1} \left(1 + \frac{g(\alpha)g(\beta)g(\gamma)}{q} T\right)}{(1-T)(1-qT)},$$

the product taken over triples of non-trivial characters

$$\alpha, \beta, \gamma: \mathbb{F}_q^\times \longrightarrow \mu_d(\mathbb{C})$$

valued in d -th order roots of unity satisfying the equality $\alpha\beta\gamma = 1$. The degree of the numerator equals $(d-1)(d-2)$, which coincides with $2g$ where g is the genus of X_d . Here for a character $\chi: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$, the $g(\chi)$ denotes the **Gauss sum**

$$g(\chi) = \sum_{x \in \mathbb{F}_q^\times} \chi(x) \exp\left(\frac{2\pi i \cdot \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)}{p}\right).$$

A key property of the Gauss sum is that $|g(\chi)| = q^{1/2}$ for $\chi \neq 1$. It follows that the $2g$ roots of $Z(X_d/\mathbb{F}_q, T)$ have absolute value $q^{-1/2}$.

21.3. The Weil conjectures

The goal of the lecture is to prove the following result, generalizing the explicit computation for the Fermat curve in Example 21.2.4.

Theorem 21.3.1 (Hasse–Schmidt–Weil, “Weil conjectures for curves”). *Let $X \subseteq \mathbb{P}^n$ be a smooth projective curve defined over \mathbb{F}_q of genus g . Then the following hold:*

(1) (Rationality) *We have $Z(X/\mathbb{F}_q, T) \in \mathbb{Q}(T)$. More precisely, the zeta function has the form*

$$Z(X/\mathbb{F}_q, T) = \frac{\prod_{i=1}^{2g} (1 - \omega_i T)}{(1-T)(1-qT)}$$

for some algebraic numbers $\omega_1, \dots, \omega_{2g}$.

(2) (Functional equation) *We have*

$$Z\left(X/\mathbb{F}_q, \frac{1}{qT}\right) = q^{1-g} T^{2-2g} Z(X/\mathbb{F}_q, T).$$

(3) (Riemann hypothesis) *The zeros of $Z(X/\mathbb{F}_q, T)$ are of absolute value $1/\sqrt{q}$. Equivalently, we have*

$$|\omega_i| = q^{1/2}, \quad i = 1, \dots, 2g.$$

Remark 21.3.2 (Weil conjectures in terms of point counting). Equivalent forms of assertions (1)–(3) expressed in terms of the numbers $\#X(\mathbb{F}_{q^r})$:

(1) $\#X(\mathbb{F}_{q^r}) = 1 + q^r - \sum_{i=1}^{2g} \omega_i^r$ for some $\omega_1, \dots, \omega_{2g} \in \overline{\mathbb{Q}}$;

(2) the ω_i come in pairs $(\omega, q/\omega)$;

$$(3) \quad |\#X(\mathbb{F}_{q^r}) - 1 - q^r| = |\sum_{i=1}^{2g} \omega_i^r| \leq 2g\sqrt{q} \text{ (“Hasse–Weil bound”)}, \text{ or } |\omega_i| = q^{1/2}.$$

We note that (1) and (2) imply that the numbers $\#X(\mathbb{F}_{q^r})$ for $r \geq g$ determine the entire sequence (see Exercise 21.3.3 below). For example, the zeta function of an elliptic curve E over \mathbb{F}_p is determined by the single number $a_p = p + 1 - \#E(\mathbb{F}_p)$.

Exercise 21.3.3. Let $\lambda_1, \dots, \lambda_g$ be complex numbers. Prove that the g numbers

$$\sum_{i=1}^g (\lambda_i^r + \lambda_i^{-r}), \quad r = 1, \dots, g$$

determine the numbers $\lambda_1, \dots, \lambda_g$ up to permutation. Deduce (setting $\lambda_i = q^{-1/2}\omega_i$) that the numbers $\#X(\mathbb{F}_{q^r})$ for $r \leq g$ determine the zeta function $Z(X/\mathbb{F}_q, T)$.

Remark 21.3.4 (General Weil conjectures). Weil famously formulated his conjectures, the analog of Theorem 21.3.1 for an arbitrary smooth projective variety X defined over \mathbb{F}_q :

(1) We have

$$Z(X/\mathbb{F}_q, T) = \frac{P_1(T)P_3(T)\dots P_{2d-1}(T)}{P_0(T)P_2(T)\dots P_{2d}(T)}, \quad d = \dim(X)$$

where $P_0(T), \dots, P_{2d}(T) \in \mathbb{Z}[T]$ are polynomials of the form

$$P_i(T) = \prod_{j=1}^{b_i} (1 - \omega_{ij}T),$$

where $P_0(T) = 1 - T$ and $P_{2d}(T) = 1 - q^d T$.

(2) We have

$$Z\left(X/\mathbb{F}_q, \frac{1}{q^d T}\right) = \pm q^{dE/2} T^E Z(X/\mathbb{F}_q, T)$$

where $E = \sum_{i=0}^{2d} (-1)^i b_i$ (the “degree” of the rational function $Z(X/\mathbb{F}_q, T)$).

(3) We have

$$|\omega_{ij}| = q^{i/2}$$

(which uniquely determines the factorization in (1)). Thus, by (2), if ω is a root of P_i , then q^d/ω is a root of P_{2d-i} .

Further, Weil conjectured a link with algebraic topology.

(4) Suppose that there exist homogeneous polynomials $f_1, \dots, f_r \in \mathbb{Z}_{(p)}[T_0, \dots, T_n]$ such that the ring $\mathbb{Z}_{(p)}[T_0, \dots, T_n]/(f_1, \dots, f_r)$ has no p -torsion and such that the projective variety defined by these equations over $\overline{\mathbb{F}}_p$ is smooth. Let $Y \subseteq \mathbb{C}P^n$ be the complex projective variety over \mathbb{C} defined by the same equations (which then will be smooth as well), equipped with the analytic topology. Then the degree $b_i = \deg P_i$ is equal the dimension of the i -th singular cohomology group $H^i(Y, \mathbb{Q})$. In particular, the integer E in (2) is equal to the Euler characteristic of Y .

These conjectures have been established by Dwork (rationality), Artin–Grothendieck (rationality, functional equation, and (4)), and Deligne (who established the Riemann hypothesis in 1973). In fact, Artin and Grothendieck developed ℓ -adic cohomology $H^i(X, \mathbb{Q}_\ell)$ of any algebraic variety and recast the conjectures in terms of eigenvalues of the Frobenius map on these cohomology groups. In this formalism, assertion (1) follows from the analog of the Lefschetz fixed point formula and (2) from Poincaré duality. Assertion (4) follows from a comparison theorem between $H^i(X, \mathbb{Q}_\ell)$ and the singular cohomology $H^i(Y, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. See [Hartshorne, Appendix C] for more details.

Remark 21.3.5 (Zeta functions of schemes over \mathbb{Z}). Let X be a scheme of finite type over \mathbb{Z} (for example, defined by a system of homogeneous equations with coefficients in $\mathbb{Z}[T_0, \dots, T_n]$). Let X_{cl} be its set of closed points (which correspond to maximal homogeneous ideals). As we have proved in the commutative algebra course, for every $x \in X_{\text{cl}}$, the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite field¹¹. We define the holomorphic function

$$\zeta_X(s) = \prod_{x \in X_{\text{cl}}} \frac{1}{1 - \#\kappa(x)^{-s}}$$

(this converges for $\text{Re}(s) > \dim(X)$). This construction has two important special cases:

- (a) If $X = \text{Spec}(\mathbb{Z})$, then X_{cl} is the set of prime numbers, and $\zeta_X(s) = \zeta(s)$ is the Riemann zeta function. More generally, for the ring of integers \mathcal{O}_K in a number field K , the zeta function of $\text{Spec}(\mathcal{O}_K)$ is the Dedekind zeta function

$$\zeta_K(s) = \prod_{0 \neq \mathfrak{p} \subseteq \mathcal{O}_K} \frac{1}{1 - \#\kappa(\mathfrak{p})^{-s}} = \sum_{0 \neq I \subseteq \mathcal{O}_K} \#(\mathcal{O}_K/I)^{-s}.$$

- (b) If $X \subseteq \mathbb{P}^n$ (over $\overline{\mathbb{F}}_q$) is a projective variety defined over \mathbb{F}_q , and $X_0 \subseteq \mathbb{P}_{\mathbb{F}_q}^n$ is the corresponding scheme over \mathbb{F}_q , then

$$\zeta_{X_0}(s) = Z(X/\mathbb{F}_q, q^{-s}).$$

To see why this is true, we need a new formula for $Z(X/\mathbb{F}_q, T)$ — see Lemma 21.4.4.

In general, we do not know whether $\zeta_X(s)$ admits a meromorphic continuation to the complex plane (this would follow from the Langlands program).

21.4. The Frobenius map

The base observation is that the field \mathbb{F}_q is the fixed-point set of the map $x \mapsto x^q$. We can use this to get some geometric understanding of $X(\mathbb{F}_q)$. The (q -th power) **Frobenius map** is the map

$$F: \mathbb{P}^n \longrightarrow \mathbb{P}^n, \quad F(x_0 : \dots : x_n) = (x_0^q : \dots : x_n^q).$$

Following a familiar notation from dynamical systems, for an endomorphism $F: X \rightarrow X$ of a set X we denote by $\text{Fix}(F) = \{x \in X : F(x) = x\}$ the set of its fixed points.

Lemma 21.4.1. *Let $X \subseteq \mathbb{P}^n$ be a locally closed subset defined over \mathbb{F}_q .*

- (a) *We have $F(X) \subseteq X$, and so F defines a map $F_X: X \rightarrow X$.*
(b) *$X(\mathbb{F}_{q^r}) = \text{Fix}(F_X^r)$.*

Proof. (a) Let $f \in \mathbb{F}_q[T_0, \dots, T_n]$ be a homogeneous polynomial. Then

$$f(T_0^q, \dots, T_n^q) = f(T_0, \dots, T_n)^q,$$

and it follows that f vanishes at a point $x \in \mathbb{P}^n$ if and only if it vanishes at $F(x)$. The assertion follows since our X is defined by vanishing and non-vanishing of a system of such polynomials f .

(b) It suffices to treat the case $X = \mathbb{P}^n$ and $r = 1$. We must show that if $0 \neq (x_0, \dots, x_n) \in \overline{\mathbb{F}}_q^{n+1}$ are such that

$$(x_0^q, \dots, x_n^q) = \lambda \cdot (x_0, \dots, x_n)$$

for some $\lambda \in \overline{\mathbb{F}}_q^\times$, then there exist $(y_0, \dots, y_n) \in \mathbb{F}_q$ and $\mu \in \overline{\mathbb{F}}_q$ such that

$$(x_0, \dots, x_n) = \mu \cdot (y_0, \dots, y_n).$$

To this end we set $\mu = \lambda^{1/(q-1)}$, which exists since $\overline{\mathbb{F}}_q$ is algebraically closed. \square

¹¹**Lemma.** *If k is a field which is of finite type over \mathbb{Z} , then k is finite. Proof.* By Chevalley's theorem (Lecture 4, Theorem 4.4.3), the image of $\text{Spec}(k) \rightarrow \text{Spec}(\mathbb{Z})$ is constructible, and hence a closed point (p) . Thus k is finitely generated over \mathbb{F}_p , and hence finite over it by Nullstellensatz (Lecture 1, Theorem 1.1.3). \square

Warning: the map $F_X : X \rightarrow X$ is bijective but not an automorphism (the extension of function fields is purely inseparable).

Definition 21.4.2. Let $X \subseteq \mathbb{P}^n$ be a locally closed subset defined over \mathbb{F}_q .

- (a) For $x \in X$, we set $\deg(x) = \min\{r : F^r(x) = x\}$ = (the smallest r such that x is \mathbb{F}_{q^r} -rational).
- (b) We set $X_0 = X/F_X$ to be the orbit space of the bijection F_X . Thus $\deg(x)$ is the size of the orbit of x .

Remark 21.4.3. Let $X_{00} \subseteq \mathbb{P}_{\mathbb{F}_q}^n$ be the “corresponding scheme over \mathbb{F}_q ” as in Remark 21.3.5 (if X is closed, then X_{00} cut out by the homogeneous ideal $\mathcal{J}(X) \cap \mathbb{F}_q[T_0, \dots, T_n]$). Then the set $X_0 = X/F_X$ in Definition 21.4.2 coincides with the set of closed points of the scheme X_{00} , and the degree $\deg(x)$ defined above coincides with the degree of the extension $[\kappa(x) : \mathbb{F}_q]$.

Lemma 21.4.4 (Euler product). *Let $X \subseteq \mathbb{P}^n$ be a locally closed subset defined over \mathbb{F}_q . We have*

$$Z(X/\mathbb{F}_q, T) = \prod_{x \in X_0} \frac{1}{1 - T^{\deg(x)}}.$$

Proof. Let $x \in X$, then $x \in \text{Fix}(F^r) = X(\mathbb{F}_{q^r})$ if and only if the order of the orbit $\deg(x)$ divides r . Counting the fixed points in two ways, we obtain

$$\#X(\mathbb{F}_{q^r}) = \sum_{x \in X_0, \deg(x)|r} \deg(x).$$

Using the formula (21.2.1) we obtain

$$\begin{aligned} Z(X/\mathbb{F}_q, T) &= \exp\left(\sum_{r \geq 1} \frac{1}{r} \left(\sum_{x \in X_0, \deg(x)|r} \deg(x)\right) T^r\right) \\ &= \exp\left(\sum_{x \in X_0} \sum_{r: \deg(x)|r} \frac{\deg(x)}{r} T^r\right) = \exp\left(\sum_{x \in X_0} \sum_{r \geq 1} \frac{1}{r} T^{r \deg(x)}\right) \\ &= \prod_{x \in X_0} \exp(\log(1 - T^{\deg(x)})) = \prod_{x \in X_0} \frac{1}{1 - T^{\deg(x)}}, \end{aligned}$$

where in the third equality we substituted $r/\deg(x)$ for r . □

Suppose now that X is a smooth curve. A divisor $D = \sum a_P P$ on X is **defined over \mathbb{F}_q** if $F(D) = D$, or equivalently if $a_P = a_{F(P)}$ for all $P \in X$. We denote by $\text{Pic}(X_0)$ the quotient of the group of divisors defined over \mathbb{F}_q by the principal divisors. We define $\text{Pic}^0(X_0)$ to be the subgroup consisting of divisor classes of degree zero. It is a finite group¹² (as will follow from the computations below), and we denote its order by h (in honor of the class number). Note that if we identify $P \in X_0$ with the sum of its orbit $P + F(P) + \dots + F^{\deg(P)-1}(P)$ then the degree of the resulting divisor equals the $\deg(x)$.

Lemma 21.4.5. *Let $X \subseteq \mathbb{P}^n$ be a smooth projective curve defined over \mathbb{F}_q .*

- (a) *We have*

$$Z(X/\mathbb{F}_q, T) = \sum_{D \geq 0 \text{ def. over } \mathbb{F}_q} T^{\deg(D)}.$$

¹²**Lemma.** $\text{Pic}^0(X_0)$ is finite. *Proof.* Let P be an effective divisor of degree $d > 2g - 2$ defined over \mathbb{F}_q . Then for every divisor D on X defined over \mathbb{F}_q , the divisor $D + P$ is of degree d and hence $\ell(D + P) > 0$ by Riemann–Roch. Thus $D + P \sim Q$ for an effective divisor Q defined over \mathbb{F}_q . Rewriting this as $D \sim Q - P$, we found a surjection from the (finite) set of all effective divisors Q of degree d defined over \mathbb{F}_q onto $\text{Pic}^0(X_0)$. □

(b) For a divisor D defined over \mathbb{F}_q , the number of effective divisors defined over \mathbb{F}_q which are linearly equivalent to D equals

$$\frac{q^{\ell(D)} - 1}{q - 1}, \quad \ell(D) = \dim H^0(X, \mathcal{O}_X(D)).$$

(c) The degree map $\text{Pic}(X_0) \rightarrow \mathbb{Z}$ is surjective.

Proof sketch. (a) Expand the Euler product in Lemma 21.4.4.

(b) These divisors are parametrized by the \mathbb{F}_q -rational points of the projective space $\mathbb{P}H^0(X, \mathcal{O}_X(D)) \simeq \mathbb{P}^{\ell(D)-1}$, which has

$$1 + q + \cdots + q^{\ell(D)-1} = \frac{q^{\ell(D)} - 1}{q - 1}$$

rational points.

(c) Omitted, see [Lorenzini, VIII 6.2]. The proof of this fact uses the zeta function (the fact that it has a simple pole at $T = 1$). Note that this is false over \mathbb{R} , for example on a real conic $C \subseteq \mathbb{P}^2$ with no rational points, every divisor defined over \mathbb{R} has even degree. Finally, note that the assertion trivially holds if $X(\mathbb{F}_q)$ is non-empty, so in particular it holds after replacing \mathbb{F}_q with \mathbb{F}_{q^r} for some $r \geq 1$. \square

21.5. Proof of Theorem 21.3.1(1): Rationality

Recall that the **Riemann–Roch theorem** on a smooth projective curve X of genus g is the formula

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g$$

valid for every divisor D on X , where K is the canonical divisor (that is, $\mathcal{O}_X(K) \simeq \omega_X$). We can pick a K which is defined over \mathbb{F}_q . We have $\ell(K) = g$ and $\deg(K) = 2g - 2$. The basic corollary of Riemann–Roch is that

$$\ell(D) = \deg(D) + 1 - g \quad \text{if } \deg(D) > 2g - 2, \quad (21.5.1)$$

as then $\deg(K - D)$ is negative and so $\ell(K - D) = 0$.

Proof of Theorem 21.3.1(1). We can now prove assertion (1) of Theorem 21.3.1 by applying the formula in Lemma 21.4.5(a) and using (21.5.1). All the sums below are over divisors or divisor classes defined over \mathbb{F}_q .

$$\begin{aligned} Z(X/\mathbb{F}_q, T) &= \sum_{D \geq 0} T^{\deg(D)} = \sum_{d \geq 0} \#\{D \geq 0 : \deg(D) = d\} T^d \\ &= \underbrace{\sum_{0 \leq d \leq 2g-2} \#\{D \geq 0 : \deg(D) = d\} T^d}_{\text{polynomial } Q(T) \text{ of degree } \leq 2g-2} + \sum_{d > 2g-2} \#\{D \geq 0 : \deg(D) = d\} T^d \\ &= Q(T) + \sum_{d > 2g-2} \sum_{\text{Pic}^d(X_0)} \frac{q^{\ell(D)} - 1}{q - 1} \\ &= Q(T) + \sum_{d > 2g-2} h \cdot \frac{q^{d+1-g} - 1}{q - 1} T^d \\ &= \frac{P(T)}{(1 - T)(1 - qT)} \end{aligned}$$

where $P(T) \in \mathbb{Z}[T]$ is a polynomial of degree $\leq 2g$ with $P(0) = 1$. (In the third line, we divided the divisors of degree $d > 2g - 2$ into linear equivalence classes. Since they are permuted by $\text{Pic}^0(X_0)$, there is h of them, and each class contributes the same number $(q^{d+1-g} - 1)/(q - 1)$ by (21.5.1), which gets us to the fourth line.) We shall prove that $\deg(P) = 2g$ in the next step. \square

21.6. Proof of Theorem 21.3.1(2): Functional equation

Proof of Theorem 21.3.1(2). The basic idea is to pair up the terms with D and $K - D$ exploiting the symmetry in the Riemann–Roch formula (Serre duality). We cannot check the formula by a simple substitution in formal power series since one side of the formula lives in $\mathbb{Q}((T))$ and the other in $\mathbb{Q}((1/T))$. So we need to be careful, and first we will treat the range of degrees from 0 to $2g - 2$ separately. The key calculation is the following. Let

$$\alpha(T) = \sum_{0 \leq \deg(D) \leq 2g-2} q^{\ell(D)} T^{\deg(D)}.$$

Then

$$T^{2g-2} q^{g-1} \alpha\left(\frac{1}{qT}\right) = \sum q^{\overbrace{\ell(D) - \deg(D) + g - 1}^{\ell(K-D)}} T^{\overbrace{2g-2 - \deg(D)}^{\deg(K-D)}} = \alpha(T).$$

Now, we can define $\beta(T)$ by

$$Z(X/\mathbb{F}_q, T) = \frac{1}{q-1} \alpha(T) + \frac{1}{q-1} \beta(T),$$

so that $Z(X/\mathbb{F}_q, T)$ satisfies the functional equation if and only if $\beta(T)$ does. Now,

$$\begin{aligned} \beta(T) &= \sum_{\deg(D) \geq 2g-1} q^{\ell(D)} T^{\deg(D)} - \sum_{\deg(D) \geq 0} T^{\deg(D)} \\ &= \sum_{\deg(D) \geq 2g-1} q^{\deg(D)+1-g} T^{\deg(D)} - \sum_{\deg(D) \geq 0} T^{\deg(D)} \\ &= h \sum_{d \geq 2g-1} q^{d+1-g} T^d - h \sum_{d \geq 0} T^d \\ &= h q^{1-g} T^{2g-1} \frac{1}{1-qT} - h \frac{1}{1-T}. \end{aligned}$$

We check easily that the operation $\phi \mapsto q^{1-g} T^{2g-2} \phi(q^{-1} T^{-1})$ interchanges the two summands above. \square

Note that since $Z(X/\mathbb{F}_q, 0) = 1$, the functional equation indeed implies that the numerator $P(T)$ has degree exactly $2g$ (and leading term q^g).

21.7. Proof of Theorem 21.3.1(3): The Riemann hypothesis

Lemma 21.7.1. *Let $\lambda_1, \dots, \lambda_m$ be complex numbers and let $\rho > 0$. The following conditions are equivalent*

- (a) $|\lambda_i| \leq \rho$ for $i = 1, \dots, m$;
- (b) $|\sum_{i=1}^m \lambda_i^r| \leq m\rho^r$ for every $r \geq 1$.

Proof. Omitted. \square

Corollary 21.7.2. *The Riemann hypothesis (3) holds if and only if the Hasse–Weil bound*

$$|\#X(\mathbb{F}_{q^r}) - 1 - q^r| \leq 2gq^{r/2}$$

is satisfied for all $r \geq 1$.

Proof. Using the formula (21.2.1), we have

$$Z(X/\mathbb{F}_q, T) = \frac{\prod_{i=1}^{2g} (1 - \omega_i T)}{(1 - T)(1 - qT)} = \exp \left(\sum \frac{1}{r} (1 + q^r - \sum_{i=1}^{2g} \omega_i^r) T^r \right),$$

from which we infer that

$$\#X(\mathbb{F}_{q^r}) = 1 + q^r - \sum_{i=1}^{2g} \omega_i^r,$$

and consequently the left-hand side of the Hasse–Weil bound equals $|\sum \omega_i^r|$. Applying Lemma 21.7.1 with $\rho = q$ and the numbers $\omega_1, \dots, \omega_{2g}$ we see that the Hasse–Weil bound holds if and only if $|\omega_i| \leq q^{1/2}$. But, by the functional equation the set $\{\omega_1, \dots, \omega_{2g}\}$ is closed under $\omega \mapsto q/\omega$. Consequently, if $|\omega_i| \leq q^{1/2}$ for all i if and only if $|\omega_i| = q^{1/2}$ for all i . \square

To finish the proof of (3), it suffices (changing the base field) to show the Hasse–Weil bound for $r = 1$. The basic idea for this is that the fixed-point set $\text{Fix}(F)$ of a map $F : X \rightarrow X$ is the intersection of the graph Γ of F with the diagonal $\Delta \subseteq X \times X$. Since our X is a curve, the product $X \times X$ is a surface, and the number of the intersection points $\#X(\mathbb{F}_q) = \#\text{Fix}(F) = \#(\Gamma \cap \Delta)$ equals the intersection number (Γ, Δ) which we may control using intersection theory on surfaces, notably the Hodge index theorem (Lecture 20, Theorem 20.3.1).

Lemma 21.7.3. *The curves Γ (graph of Frobenius) and Δ (diagonal) on $X \times X$ intersect transversely. Consequently,*

$$(\Gamma, \Delta) = \#X(\mathbb{F}_q).$$

Proof. Since both curves are smooth (they are isomorphic to X via the first projection $X \times X \rightarrow X$) it suffices to show that for every $(x, x) \in \Gamma \cap \Delta$, the tangent lines $T_{(x,x)}\Gamma$ and $T_{(x,x)}\Delta$ are distinct. Both are subspaces of the tangent space $T_{(x,x)}X \times X = T_x X \oplus T_x X$. The second one $T_{(x,x)}\Delta$ is simply the diagonal $T_x X$. We claim that $T_{(x,x)}\Gamma = T_x X \oplus 0$, which follows from the fact that since the Frobenius map is purely inseparable, its derivative vanishes everywhere. Since we have not discussed inseparable morphisms, we do a local computation on \mathbb{P}^n . In an affine open $U \simeq \mathbb{A}^n \subseteq \mathbb{P}^n$ containing x , the map $\text{id} \times F : U \rightarrow U \times U$ is given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x_1^q, \dots, x_n^q) = (x_1, \dots, x_n, y_1, \dots, y_n).$$

Its image (the graph of F_U) is therefore cut out by the equations $y_i - x_i^q = 0$. Now, $dx_i^q = qx_i^{q-1} dx_i = 0$. Therefore the cotangent space is spanned by dy_1, \dots, dy_n , and hence the tangent space is $U \oplus 0$. The tangent space to the graph Γ of F_x is thus contained in $T_x \mathbb{P}^n \oplus 0$, and hence equal to $T_x X \oplus 0$. \square

Lemma 21.7.4. *Let $x_0 \in X$. Let $C_v = \{x_0\} \times X$ and $C_h = X \times \{x_0\}$ be the “horizontal” and “vertical” axes through $(x_0, x_0) \in X \times X$. Then*

$$\begin{aligned} a) \quad (\Delta, \Delta) &= 2 - 2g, & b) \quad (\Gamma, \Gamma) &= q(2 - 2g), & c) \quad (\Gamma, \Delta) &= \#X(\mathbb{F}_q), \\ d) \quad (\Delta, C_v) &= (\Delta, C_h) = (\Gamma, C_v) = 1, & e) \quad (\Gamma, C_h) &= q. \end{aligned}$$

Proof. a) Follows from the adjunction formula (Lecture 19, Example 19.4.5). To show b), notice that $\Gamma = (F \times \text{id})^*(\Delta)$ and $F \times \text{id}$ is a finite flat morphism of degree q , so that the assertion follows from Lecture 19, Example 19.4.6. Assertion c) is the previous lemma. Formulas in d) are obvious (the intersections have only one point at which the tangent lines are distinct). Formula e) follows from the fact that $(\Gamma, C_h) = \deg(F^*(\{x_0\})) = q$. \square

We proved the following result last time, as a corollary of the Hodge index theorem.

Proposition 21.7.5 (Lecture 20, Proposition 20.3.7). *For any divisors D_1 and D_2 on $X \times X$ we have*

$$|(C_h \cdot D_1)(C_v \cdot D_2) + (C_h \cdot D_2)(C_v \cdot D_1) - (D_1 \cdot D_2)| \leq \sqrt{2(C_h \cdot D_1)(C_v \cdot D_1) - (D_1 \cdot D_1)} \\ \cdot \sqrt{2(C_h \cdot D_2)(C_v \cdot D_2) - (D_2 \cdot D_2)}.$$

Proof. We work in the space $\text{NS}(X)_{\mathbb{R}}$ of numerical equivalence class of divisors with real coefficients. Consider the quadratic form

$$\lambda(D) = 2(C_h \cdot D)(C_v \cdot D) - (D \cdot D)$$

and the corresponding symmetric bilinear form

$$\lambda(D_1, D_2) = \frac{\lambda(D_1 + D_2) - \lambda(D_1) - \lambda(D_2)}{2} = (C_h \cdot D_1)(C_v \cdot D_2) + (C_h \cdot D_2)(C_v \cdot D_1) - (D_1 \cdot D_2).$$

Then the inequality reads

$$|\lambda(D_1, D_2)| \leq \sqrt{\lambda(D_1)\lambda(D_2)}.$$

This looks like the Cauchy–Schwarz inequality, and will follow if we show that $\lambda(D) \geq 0$ for all D .

Consider the subspace of $\text{NS}(X)_{\mathbb{R}}$ spanned by the vectors $D_1 = C_h$, $D_2 = C_v$, and $D_3 = D$. Consider the determinant

$$\det[(D_i \cdot D_j)] = \begin{vmatrix} 0 & 1 & (C_h \cdot D) \\ 1 & 0 & (C_v \cdot D) \\ (C_h \cdot D) & (C_v \cdot D) & (D \cdot D) \end{vmatrix} = \lambda(D)$$

This is either zero (if D_1 , D_2 , and D_3 give linearly dependent elements of $\text{NS}(X)_{\mathbb{R}}$), or positive otherwise (indeed, then the span of the D_i is a three-dimensional vector space on which the intersection form has signature $(1, 2)$ since $(D_1 + D_2)^2 = 2 > 0$). \square

Proof of Theorem 21.3.1(3). We apply Proposition 21.7.5 to $D_1 = \Gamma$ and $D_2 = \Delta$. Lo and behold:

$$\left| \underbrace{(C_h \cdot \Delta)}_q \underbrace{(C_v \cdot \Gamma)}_1 + \underbrace{(C_h \cdot \Gamma)}_1 \underbrace{(C_v \cdot \Delta)}_1 - \underbrace{(\Delta \cdot \Gamma)}_{\#X(\mathbb{F}_q)} \right| \leq \sqrt{2 \underbrace{(C_h \cdot \Gamma)}_q \underbrace{(C_v \cdot \Gamma)}_1 - \underbrace{(\Gamma \cdot \Gamma)}_{q(2-2g)}} \times \sqrt{2 \underbrace{(C_h \cdot \Delta)}_1 \underbrace{(C_v \cdot \Delta)}_1 - \underbrace{(\Delta \cdot \Delta)}_{2-2g}} = 2g\sqrt{q}.$$

\square

A. Homework problems

Problems marked with an asterisk are for extra credit.

A.1. Problem Set 1

In all of the problems, k denotes an algebraically closed field.

Problem 1.1. Suppose that $\text{char}(k) \neq 2$, let $f \in k[X]$ be a polynomial of degree ≥ 1 . Show that the plane curve

$$Y = \{(x, y) : y^2 = f(x)\} \subseteq k^2$$

is irreducible, and that it is smooth if and only if $f(X)$ has no repeated roots. Show that

$$Y = \{(x, y) : y^2 = x^3 + ax + b\} \subseteq k^2$$

is smooth if and only if $4a^3 + 27b^2 \neq 0$.

Problem 1.2 (The twisted cubic curve, Hartshorne (I, Ex. 1.2)). Show that

$$Y = \{(t, t^2, t^3) : t \in k\} \subseteq k^3$$

is an irreducible algebraic set of dimension 1. Find generators for the ideal $\mathcal{J}(Y)$. Show that $k[X, Y, Z]/(\mathcal{J}(Y))$ is isomorphic to a polynomial ring in one variable over k .

Problem 1.3 (cf. Hartshorne (I, Ex. 1.7)). Let X be a Noetherian topological space.

- Show that every subset $Z \subseteq X$ is Noetherian in its induced topology.
- Show that X is quasi-compact (every open cover has a finite subcover).
- Show that if X is also Hausdorff, then it is finite and discrete.

Problem 1.4 (cf. Hartshorne (I, Ex. 1.10)). Let X be a topological space.

- If $Y \subseteq X$ is a subspace, then $\dim(Y) \leq \dim(X)$.
- Give an example of a topological space X and a dense open subset $U \subseteq X$ with $\dim(U) < \dim(X)$.
- Suppose that X is irreducible and $\dim(X) < \infty$. If $Y \subseteq X$ is a closed subset with $\dim(Y) = \dim(X)$, then $Y = X$.

Problem 1.5 (Hartshorne (I, Ex. 3.7a)). Show that every two curves in the projective plane \mathbb{P}^2 intersect. In other words, for every pair $f, g \in k[X, Y, Z]$ of non-constant homogeneous polynomials, the system

$$\begin{cases} f(X, Y, Z) = 0, \\ g(X, Y, Z) = 0 \end{cases}$$

has a solution $(x, y, z) \neq (0, 0, 0)$.

★ **Problem 1.6.** Are the spaces \mathbb{C}^2 and $\mathbb{C}^2 \setminus \{(0, 0)\}$ homeomorphic when equipped with the Zariski topology?

A.2. Problem Set 2

Problem 2.1. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set endowed with an action of a finite group G . Let $A = \mathcal{O}(X)$ be the corresponding algebra, so that G acts on A by k -algebra automorphisms. Let

$$B = A^G = \{f \in A : g * f = f \text{ for all } g \in G\} \subseteq A$$

be the subring of invariants. Last semester, we deduced from the Artin–Tate lemma that B is a finitely generated k -algebra. This allows us to consider $Y = \text{MSpec}(B)$, the affine algebraic set with $\mathcal{O}(Y) = B$. Let $\pi: X \rightarrow Y$ be the morphism induced by the inclusion $B \hookrightarrow A$. Show that

- (a) $Y = X/G$ is the orbit space (that is, $\pi(x) = \pi'(x')$ iff $x = g(x')$ for some $g \in G$);
- (b) for every morphism $f: X \rightarrow Z$ to an affine algebraic set Z such that $f \circ g = f$ for every $g \in G$ there exists a unique $\bar{f}: Y \rightarrow Z$ such that $f = \bar{f} \circ \pi$.

Hint for (a): First show that for any two disjoint closed subsets $Z_0, Z_1 \subseteq X$ there exists an $f \in A$ such that $f(z) = 0$ for all $z \in Z_0$ and $f(z) = 1$ for all $z \in Z_1$. Apply this to a pair of G -orbits, then use the G -action to obtain elements of A^G .

Problem 2.2. Let $k = \mathbb{C}$. For which values of the parameter $\lambda \in k$ is the hypersurface $X \subseteq \mathbb{P}^n$ defined by

$$\lambda(X_0^{n+1} + \dots + X_n^{n+1}) = (n+1)X_0 \cdots X_n$$

nonsingular?

Info: This is the Dwork family of Calabi–Yau hypersurfaces of dimension $n - 1$. The case $n = 3$ (quintic threefolds) has been extensively studied in the context of mirror symmetry.

Problem 2.3. Let X be a variety and let $X_1, \dots, X_n \subseteq X$ be constructible subsets such that $X = X_1 \cup \dots \cup X_n$. Show that one of the X_i contains a non-empty open subset of X .

Problem 2.4. Let $X = V(f) \subseteq \mathbb{P}^2$ be a conic (so $f \in k[X, Y, Z]$ is an irreducible homogeneous polynomial of degree two). Show that $X \simeq \mathbb{P}^1$.

Problem 2.5. Assume that $\text{char}(k) \neq 2$. Let $X = V(f) \subseteq \mathbb{P}^3$ be an irreducible quadric surface (so $f \in k[T_0, \dots, T_3]$ is an irreducible homogeneous polynomial of degree two). Show that X is rational (i.e., has a non-empty open subset isomorphic to an open subset of \mathbb{P}^2).

Hint: first show that up to a linear change of coordinates, f can be put in the form $T_0T_1 - T_2T_3$ or $T_0T_1 - T_2^2$.

★ **Problem 2.6.** Suppose that k is uncountable. Let X be a variety and let $X_n \subseteq X$ ($n \geq 1$) be a sequence of constructible subsets such that $X = \bigcup_{n=1}^{\infty} X_n$. Show that one of the X_n contains a non-empty open subset of X .

A.3. Problem Set 3

Algebraic groups

Problem 3.1. Let G be an algebraic group. Show that G is

- (a) separated,
- (b) non-singular.

Hint for (a): Use the multiplication map $\mu: G \times G \rightarrow G$ to describe the diagonal.

Problem 3.2. Let $G = \mathbb{G}_m^a$ and $H = \mathbb{G}_m^b$.

- (a) Prove that every map $f: G \rightarrow H$ with $f(1) = 1$ is a group homomorphism.
- (b) Prove that group homomorphisms $G \rightarrow H$ form a group isomorphic to $\text{Hom}(\mathbb{Z}^b, \mathbb{Z}^a)$.

Blow-ups

Problem 3.3 (cf. Hartshorne Ex. I 5.7). Let $Y \subseteq \mathbb{P}^2$ be a nonsingular projective plane curve defined by the homogeneous equation $f(x, y, z) = 0$ (for $f \in k[x, y, z]$ square-free, homogeneous of degree $d > 1$). Let $X = V(f) \subseteq \mathbb{A}^3$ be the “affine cone” of Y .

- (a) Show that the origin $0 \in \mathbb{A}^3$ is the unique singular point of X .
- (b) Show that the strict transform \tilde{X} in the blowup of \mathbb{A}^3 at the origin is non-singular.
- (c) Show that the tangent cone $\tilde{X} \cap \pi^{-1}(0) \subseteq \mathbb{P}^2$ is equal to Y .

Sheaves

Problem 3.4 (Flasque/flabby sheaves, cf. Hartshorne Ex. II 1.16). A sheaf \mathcal{F} on a topological space X is called *flasque* (a.k.a. *flabby*) if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective.

- (a) Show that a constant sheaf on an irreducible topological space is flasque.
- (b) Suppose that $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of sheaves and that \mathcal{F}' is flasque. Show that for every open $U \subseteq X$, the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

is exact.

- (c) Let $f: Y \rightarrow X$ be a continuous map and let \mathcal{F} be a flasque sheaf on Y . Show that the sheaf $f_*\mathcal{F}$ on X is flasque.
- (d) For any sheaf \mathcal{F} , the sheaf $D(\mathcal{F})$ of discontinuous sections of \mathcal{F} is flasque.

Problem 3.5 (cf. Hartshorne Ex. II 1.21). Let X be an algebraic set and let \mathcal{O}_X be its sheaf of regular functions. Let $Y \subseteq X$ be a closed subset and let $i: Y \rightarrow X$ be the inclusion.

- (a) For each open subset $U \subseteq X$, let $\mathcal{J}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{J}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* \mathcal{J}_Y of Y , and it is a subsheaf of the sheaf of rings \mathcal{O}_X .
- (b) Show that the quotient sheaf $\mathcal{O}_X/\mathcal{J}_Y$ is isomorphic to $i_*\mathcal{O}_Y$, so that we have a short exact sequence

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0.$$

- (c) Consider the case $X = \mathbb{P}^1$ and $Y = \{x, y\}$ for distinct points $x, y \in X$. Show that $\mathcal{O}_X(X) \rightarrow (i_*\mathcal{O}_Y)(X)$ is not surjective (even though $\mathcal{O}_X \rightarrow i_*\mathcal{O}_Y$ is surjective by (b)).

A.4. Problem Set 4

In this class, we are primarily interested in algebraic sets and varieties. The exercises below are formulated for schemes, but feel free to replace the word *scheme* with *algebraic set* and each Spec with MSpec (the proofs should be the same).

In the problems below, use Problem 1 to solve Problem 2, and Problems 2 and 3 to solve Problem 4.

Recall that an open subset $U \subseteq X$ of an affine scheme X is called **distinguished** if $U = D(f) \simeq \text{Spec}(\mathcal{O}(X)[f^{-1}])$ for some $f \in \mathcal{O}(X)$.

Problem 4.1. Let X be a scheme. Let $U, V \subseteq X$ be two affine open subsets. Prove that every $x \in U \cap V$ admits an open neighborhood $W \subseteq U \cap V$ which is distinguished in both U and V .

Problem 4.2 (Affine communication lemma). Let X be a scheme and let \mathcal{U} be a family of affine open subsets of X enjoying the following property: for every affine open $U \subseteq X$ and every finite cover $U = U_1 \cup \dots \cup U_r$ by distinguished opens $U_1, \dots, U_r \subseteq U$, we have

$$U \in \mathcal{U} \quad \Leftrightarrow \quad U_1, \dots, U_r \in \mathcal{U}.$$

Suppose that X admits an affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ with $U_\alpha \in \mathcal{U}$. Show that every affine open $U \subseteq X$ belongs to \mathcal{U} .

Problem 4.3. Let X be either a scheme or an algebraic set. Let $f_1, \dots, f_r \in \mathcal{O}(X)$ be elements such that the opens $D(f_1), \dots, D(f_r) \subseteq X$ are affine and which generate the unit ideal in $\mathcal{O}(X)$. Show that X is affine. *Hint:* We have a map $\tau: X \rightarrow \text{Spec}(\mathcal{O}(X))$ (corresponding to the identity $\mathcal{O}(X) \rightarrow \mathcal{O}(X)$). Show that it is an isomorphism.

Problem 4.4. Let $f: Y \rightarrow X$ be a map of schemes. Prove that f is affine if and only if there exists an affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that $f^{-1}(U_\alpha)$ is affine for each $\alpha \in I$.

Problem 4.5. Let X be a noetherian scheme and \mathcal{F} a coherent sheaf on X . Prove that \mathcal{F} is locally free of rank one if and only if there exists a coherent sheaf \mathcal{M} such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M} \simeq \mathcal{O}_X$. Such \mathcal{O}_X -modules are called **invertible**.

★ **Problem 4.6.** Let X be an algebraic set. Let $f_1, \dots, f_r \in \mathcal{O}(X)$ be elements such that the opens $D(f_1), \dots, D(f_r) \subseteq X$ are affine and cover X . Does that imply that X is quasi-affine?

A.5. Problem Set 5

Problem 5.1. Let $n \geq 1$. Does there exist a surjective map $\mathbb{A}^n \rightarrow \mathbb{P}^n$?

Problem 5.2. Compute the zeta function $\zeta_{\mathbb{P}^n}(s)$ of the projective space \mathbb{P}^n over a finite field \mathbb{F}_q .

Problem 5.3. Compute $H^1(S^1, \mathbb{Z})$ directly from our definition of cohomology. (Here S^1 is the circle.)

Problem 5.4. Let X be a noetherian topological space and let $\{\mathcal{F}_\alpha\}$ be an inductive system of abelian sheaves on X . Show that the presheaf $U \mapsto \varinjlim \mathcal{F}_\alpha(U)$ is a sheaf, denoted by $\varinjlim \mathcal{F}_\alpha$. In particular, the natural map

$$\varinjlim \Gamma(X, \mathcal{F}_\alpha) \longrightarrow \Gamma(X, \varinjlim \mathcal{F}_\alpha)$$

is an isomorphism.

Problem 5.5. Let D be a prime divisor on a smooth variety X . Show that $\text{Pic}(X \setminus D) \simeq \text{Pic}(X) / \langle \mathcal{O}_X(D) \rangle$ where $\langle \mathcal{O}_X(D) \rangle = \{ \mathcal{O}_X(nD), n \in \mathbb{Z} \}$ is the subgroup generated by the class of $\mathcal{O}_X(D)$. Use this to compute $\text{Pic}(U)$ where $U = D(f) \subseteq \mathbb{P}^2$ is the complement of an irreducible curve of degree d .

★ **Problem 5.6.** Let $V \simeq \mathbb{R}^n$ be a real vector space endowed with a symmetric bilinear form $\langle x, y \rangle$ of signature $(1, n-1)$. Let $x_1, x_2 \in V$ be two vectors such that

$$\langle x_1, x_1 \rangle = 0 = \langle x_2, x_2 \rangle, \quad \langle x_1, x_2 \rangle = 1.$$

Show that for every $y_1, y_2 \in V$ the following inequality holds

$$|\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle + \langle x_1, y_2 \rangle \cdot \langle x_2, y_1 \rangle - \langle y_1, y_2 \rangle| \leq \sqrt{2\langle x_1, y_1 \rangle \cdot \langle x_2, y_1 \rangle - \langle y_1, y_1 \rangle} \cdot \sqrt{2\langle x_1, y_2 \rangle \cdot \langle x_2, y_2 \rangle - \langle y_2, y_2 \rangle}.$$

Hint: Apply Cauchy–Schwarz to the quadratic form $q(y) = 2\langle x_1, y \rangle \cdot \langle x_2, y \rangle - \langle y, y \rangle$.

B. Summary of commutative algebra

This document lists the bare definitions and facts we need for our algebraic geometry course. In case you haven't studied some of this material, it should serve as a guideline for what you need to learn.

B.1. Commutative rings

An **commutative ring** (or simply a **ring**) is an abelian group $A = (A, +, 0)$ together with a bilinear associative, commutative, and unital multiplication $\cdot : A \times A \rightarrow A$, that is: we have an element $1 \in A$ and

$$(x + y) \cdot (x' + y') = xx' + xy' + yx' + yy', \quad (x \cdot y) \cdot z = x \cdot (y \cdot z), \quad x \cdot y = y \cdot x, \quad 1 \cdot x = x$$

for all $x, x', y, y', z \in A$. A **homomorphism** (or morphism, or simply map) of rings is a homomorphism of abelian groups $\phi : A \rightarrow B$ such that $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ and $\phi(1) = 1$ (the latter condition is not automatic!). Rings form a category which we denote by **CAlg** (for “commutative algebras”). The ring \mathbb{Z} is the initial object of **CAlg**: for every A , there exists a unique map $\mathbb{Z} \rightarrow A$. The zero ring 0 is the final object, and if $0 \rightarrow A$ is a map, then $A = 0$.

If $\{A_\alpha\}_{\alpha \in I}$ is a family of rings, then the cartesian **product** $A = \prod_{\alpha \in I} A_\alpha$ with coordinate-wise multiplication and addition is a ring, and the projection maps $A \rightarrow A_\alpha$ are homomorphisms (A is the categorical product of $\{A_\alpha\}$ in **CAlg**).

For a ring A , an **A -algebra** (or: algebra over A) is a ring B together with a homomorphism $A \rightarrow B$ (which we often neglect to name), and a morphism of A -algebras $B \rightarrow C$ is a map of rings $B \rightarrow C$ such that the triangle

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow \\ A & & C \\ & \searrow & \\ & & \end{array}$$

commutes. We denote the category of A -algebras by **CAlg_A**. Then A (meant as the identity $A \rightarrow A$) is the initial object of **CAlg_A**. We have **CAlg** = **CAlg_ℤ**.

A **subring** of A is a subgroup $B \subseteq A$ containing 1 and closed under multiplication, or which is the same an injective ring map $B \rightarrow A$.

An **ideal** in a ring A is a subgroup $I \subseteq A$ such that $A \cdot I = I$ (that is, if $x \in A$ and $y \in I$, then $xy \in I$). There is a unique ring structure on the quotient group A/I making the quotient map $A \rightarrow A/I$ a ring homomorphism. For a map of rings $\phi : A \rightarrow B$, the kernel $\ker(\phi) \subseteq A$ is an ideal, the image $\text{im}(\phi) \subseteq B$ is a subring, and we have $A/\ker(\phi) \simeq \text{im}(\phi)$. For an ideal $I \subseteq A$, a map $\phi : A \rightarrow B$ factors (uniquely) through A/I if and only if $I \subseteq \ker(\phi)$. If $\phi : A \rightarrow B$ is a map and $J \subseteq B$ is an ideal, then $\phi^{-1}(J)$ is an ideal. Ideals in A/I are in bijection with ideals of A containing I . For a subset $I_0 \subseteq A$, the set $I = (I_0)$ of all linear combinations $a_1x_1 + \dots + a_nx_n$ with $a_i \in A$ and $x_i \in I_0$ is the smallest ideal of A containing I_0 , called the ideal generated by I_0 . For $I_0 = \{x_1, \dots, x_n\}$, we write $(I_0) = (x_1, \dots, x_n)$. An ideal of the form $I = (f)$ for a single $f \in A$ is called **principal**. For a map $\phi : A \rightarrow B$ and an ideal $I \subseteq A$, we write $I \cdot B$ for the ideal of B generated by $\phi(I)$. For a family of ideals $\{I_\alpha\}$, the intersection $\bigcap I_\alpha$ is an ideal, and we denote by $\sum I_\alpha$ the ideal generated by $\bigcup I_\alpha$. For two ideals $I, J \subseteq A$, the product $I \cdot J$ is the ideal generated by $\{xy : x \in I, y \in J\}$.

An ideal $\mathfrak{p} \subseteq A$ is **prime** if its complement $A \setminus \mathfrak{p}$ is a monoid (unital semigroup), or equivalently if $\mathfrak{p} \neq A$ and $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. If $\phi : A \rightarrow B$ is a map and $\mathfrak{q} \subseteq B$ is a prime ideal, then $\phi^{-1}(\mathfrak{q}) \subseteq A$ is a prime ideal.

An element $x \in A$ is a **unit** (or invertible) if $xy = 1$ for some $y \in A$. Such a y is unique and is denoted by x^{-1} . Units of a ring A form a group under multiplication, denoted by A^\times . A ring A is a **field** if $A^\times = A \setminus \{0\}$ (in particular the zero ring is not a field). A map $A \rightarrow B$ induces a map $A^\times \rightarrow B^\times$.

A nonzero ring A is a **domain** if it has no zerodivisors, i.e. if $xy = 0$ implies $x = 0$ or $y = 0$. Thus A is a domain if and only if $(0) \subseteq A$ is a prime ideal. An ideal $I \subseteq A$ is prime if and only if A/I is a domain.

An ideal $\mathfrak{m} \subseteq A$ is **maximal** if $\mathfrak{m} \neq A$ and which is maximal with respect to this property. Equivalently, the quotient A/\mathfrak{m} is a field, and every maximal ideal is a prime ideal.

Proposition B.1.1. *Every nonzero ring admits a maximal ideal.*

A ring A is **local** if it has a unique maximal ideal \mathfrak{m}_A , or equivalently if its non-units form an ideal (which is then maximal). Its **residue field** is the quotient $k_A = A/\mathfrak{m}_A$. A homomorphism $\phi: A \rightarrow B$ between local rings is **local** if $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$. It then induces a homomorphism $k_A \rightarrow k_B$ between residue fields.

An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n \geq 1$. The set of all nilpotent elements of A is an ideal called its **nilradical** and denoted by $\sqrt{0}$. More generally, for an ideal $I \subseteq A$ the set of all elements $x \in A$ such that $x^n \in I$ for some $n \geq 1$ is an ideal called the **radical** of I and denoted by \sqrt{I} . We say that the ideal I is **radical** if $I = \sqrt{I}$. We say that A is **reduced** if it has no nonzero nilpotent elements, i.e. if (0) is a radical ideal.

Proposition B.1.2. *For an ideal $I \subseteq A$, we have $\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$ (the intersection of all prime ideals containing I). In particular, an element $x \in A$ is nilpotent if and only if x belongs to every prime ideal of A .*

For a ring A and a (possibly infinite) set S , we denote by $A[S]$ the **polynomial ring** over A in the set of variables T_s for all $s \in S$. We use $A[T_1, \dots, T_n]$ as a shorthand for $A[\{1, \dots, n\}]$. This ring has the universal property: for any A -algebra B , giving a map of A -algebras $A[S] \rightarrow B$ is the same as giving a map of sets $\gamma: S \rightarrow B$. An A -algebra B is **finitely generated** (or “of finite type”) if there exists an A -algebra surjection $A[S] \rightarrow B$ for a finite set S .

For an A -algebra B (i.e. a map $A \rightarrow B$), a **presentation** of B over A is a triple (S, γ, R) where S is a set, $\gamma: S \rightarrow B$ is a map of sets such that the corresponding map $\tilde{\gamma}: A[S] \rightarrow B$ is surjective, and $R \subseteq A[S]$ is a set which generates $\ker(\tilde{\gamma})$ as an ideal. Thus $B \simeq A[S]/(R)$. We say that B is **finitely presented** over A if it admits a presentation (S, γ, R) where both S and R are finite. If $S = \{1, \dots, n\}$ and $R = \{f_1, \dots, f_r\}$, we write

$$B = A[T_1, \dots, T_n]/(f_1, \dots, f_r)$$

for the quotient $A[S]/(R)$. Thus a finitely presented A -algebra is one isomorphic to a quotient of the above type. The above quotient has the following universal property: giving an A -algebra map $B \rightarrow C$ is the same as giving elements $t_1, \dots, t_n \in C$ such that $f_i(t_1, \dots, t_n) = 0$ for $i = 1, \dots, r$. Here $f_i(t_1, \dots, t_n)$ is obtained by substituting formally the elements $t_i \in C$ for the variables T_i .

A subset $S \subseteq A$ is a **multiplicative system** if it contains 1 and is closed under multiplication. The **localization** $A[S^{-1}]$ is the ring consisting of fraction symbols a/s where $a \in A$ and $s \in S$, where $a/s = a'/s'$ if $t(as' - a's) = 0$ for some $t \in S$, with the usual rules for multiplication and addition. Thus $(1/s)(s/1) = 1$ and hence $s/1$ is a unit in $A[S^{-1}]$. The map $A \rightarrow A[S^{-1}]$ sending a to $a/1$ is a ring homomorphism and has the following universal property: a map $\phi: A \rightarrow B$ factors (uniquely) through $A[S^{-1}]$ if and only if $\phi(S) \subseteq B^\times$. Key examples:

- For $f \in A$, the set of powers $S = \{1, f, f^2, \dots\}$ is a multiplicative system and $A[S^{-1}]$ is denoted more succinctly by $A[f^{-1}]$ (we have $A[f^{-1}] \simeq A[T]/(fT - 1)$, in particular it is finitely presented over A).
- If $\mathfrak{p} \subseteq A$ is a prime ideal, then $A \setminus \mathfrak{p}$ is a multiplicative system, and we denote $A[(A \setminus \mathfrak{p})^{-1}]$ by $A_{\mathfrak{p}}$ and call it the localization at \mathfrak{p} . It is a local ring with unique maximal ideal $\mathfrak{p} \cdot A_{\mathfrak{p}}$.
- In the special case $\mathfrak{p} = (0)$ in a domain A , the ring $A_{(0)}$ is a field, called the **fraction field** of A and denoted by $\text{Frac}(A)$.

In general, the kernel of $A \rightarrow A[S^{-1}]$ consists of all $x \in A$ such that $xy = 0$ for some $y \in S$, and prime ideals in $A[S^{-1}]$ correspond to prime ideals of A which are disjoint from S . For a prime ideal $\mathfrak{p} \subseteq A$, we denote by $\kappa(\mathfrak{p})$ the residue field of the local ring $A_{\mathfrak{p}}$, or equivalently the fraction field of the quotient A/\mathfrak{p} (we call it the **residue field** of \mathfrak{p}).

B.2. The spectrum

For a ring A , we denote by $\text{Spec}(A)$ the set of all prime ideals of A , called the **spectrum** of A . However, we treat its elements (points) as independent beings, and for $x \in \text{Spec}(A)$ we write \mathfrak{p}_x for the “corresponding prime ideal.” For $x \in \text{Spec}(A)$, we denote by $\kappa(x)$ the residue field $\kappa(\mathfrak{p}_x)$, and for $f \in A$ we write $f(x)$ for the image of f in $\kappa(x)$. Thus $f \in A$ defines a “field-valued function” on $\text{Spec}(A)$, where the codomain $\kappa(x)$ depends on the point x . Then $f \in A$ is a unit if and only if $f(x) \neq 0$ for all $x \in \text{Spec}(A)$, and nilpotent if and only if $f(x) = 0$ for all $x \in \text{Spec}(A)$.

For $f \in A$, we write

$$D(f) = \{x \in \text{Spec}(A) : f(x) \neq 0\} \subseteq \text{Spec}(A)$$

(in terms of prime ideals, this is the set of primes containing f). We have $D(fg) = D(f) \cap D(g)$. We give $\text{Spec}(A)$ the topology generated by these sets; thus $Z \subseteq \text{Spec}(A)$ is closed if and only if $Z = V(I)$ for some ideal $I \subseteq A$, where

$$V(I) = \{x \in \text{Spec}(A) : f(x) = 0 \text{ for all } f \in I\} \subseteq \text{Spec}(A).$$

The construction $I \mapsto V(I)$ defines a bijection between radical ideals of A and closed subsets of $\text{Spec}(A)$.

The space $\text{Spec}(A)$ is T_0 (for $x \neq y$ we can find an open subset containing exactly one of x, y) but typically not Hausdorff. It is **quasi-compact** (every open cover has a finite subcover), and so are its base open subsets $D(f) \simeq \text{Spec}(A[f^{-1}])$.

An element $x \in A$ is **idempotent** if $x^2 = x$; then $y = 1 - x$ is also idempotent, and $A \simeq A/(x) \times A/(y)$. The space $\text{Spec}(A)$ is the disjoint union of $V(x) = \text{Spec}(A/(x)) = D(y)$ and $V(y) = \text{Spec}(A/(y)) = D(x)$. Conversely, if $Z \subseteq \text{Spec}(A)$ is a clopen subset, then $Z = V(x)$ for an idempotent x .

On a topological space X , a point x is a **specialization** of a point y (we write $y \rightsquigarrow x$) if x belongs to the closure of $\{y\}$. On $\text{Spec}(A)$, we have $y \rightsquigarrow x$ if and only if $\mathfrak{p}_y \subseteq \mathfrak{p}_x$. In particular, closed points are those whose corresponding prime ideal is maximal. The set of all closed points of $\text{Spec}(A)$ is denoted by $\text{MSpec}(A)$ and called the **maximal spectrum** of A . If A is a domain, we denote by $\eta \in \text{Spec}(A)$ the point corresponding to the prime ideal (0) , and call it the **generic point** of $\text{Spec}(A)$. It specializes to every other point.

A ring map $\phi^* : A \rightarrow B$ induces a continuous map $\phi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ (here we use the geometer’s notation, treating Spec as the primary object). In the special case of the quotient $A \rightarrow A/I = B$, this map is a homeomorphism of $\text{Spec}(A/I)$ onto $V(I)$. For the map $A \rightarrow A[f^{-1}] = B$, the map is a homeomorphism of $\text{Spec}(A[f^{-1}])$ onto $D(f)$. (Thus we think of A/I as “functions on $V(I)$ ” and of $A[f^{-1}]$ as “functions on $D(f)$ ” — we will make this precise later using sheaves.) For a domain A , a map $A \rightarrow B$ is injective if and only if the generic point $\eta \in \text{Spec}(A)$ is in the image of $\text{Spec}(A) \rightarrow \text{Spec}(B)$.

For a map $\phi^* : A \rightarrow B$ and $x \in \text{Spec}(A)$, the **fiber** $\phi^{-1}(x) \subseteq \text{Spec}(B)$ is naturally homeomorphic to $\text{Spec}(B \otimes_A \kappa(x))$ (see the next section for the definition of \otimes).

A topological space X is **irreducible** if it is nonempty and cannot be expressed as the union $X = Y_0 \cup Y_1$ of two proper closed subsets $Y_i \subseteq X$. The spectrum $\text{Spec}(A)$ is irreducible if and only if $\sqrt{0}$ is a prime ideal (for example, if A is a domain), in which case the corresponding point $\eta \in \text{Spec}(A)$ is the unique generic point (meaning that $\{\eta\}$ is dense). More generally, for any A , every irreducible closed subset $Z \subseteq \text{Spec}(A)$ is of the form $\overline{\{x\}}$ for a unique point $x \in \text{Spec}(A)$ (in fact, $Z = V(\mathfrak{p}_x) = \text{Spec}(A/\mathfrak{p}_x)$).

A topological space X is **Noetherian** if every increasing sequence of open subsets stabilizes, or equivalently if every open subset $U \subseteq X$ is quasi-compact. For example $\text{Spec}(A)$ is Noetherian if A is a Noetherian ring (see below for the definition of a Noetherian ring). A Noetherian topological space can

be written uniquely as a finite union of irreducible closed subsets (called its **irreducible components**) $X = Z_1 \cup \dots \cup Z_r$ such that Z_i is not contained in Z_j for $i \neq j$.

B.3. Modules

A **module** M over a ring A is an abelian group $(M, 0, +)$ together with a map $\cdot : A \times M \rightarrow M$ which is unital, bilinear, and associative in the sense that

$$1 \cdot m = m, \quad (x + x') \cdot (m + m') = x \cdot m + x' \cdot m + x \cdot m' + x' \cdot m', \quad (x \cdot x') \cdot m = x \cdot (x' \cdot m)$$

for $x, x' \in A$ and $m, m' \in M$ (note that the first \cdot in the third formula denotes multiplication in A). Modules over a field k are precisely the k -vector spaces. A morphism from an A -module M to an A -module N is a homomorphism of abelian groups $\phi : M \rightarrow N$ such that $\phi(x \cdot m) = x \cdot \phi(m)$. Modules over A form a category denoted by \mathbf{Mod}_A .

The ring A is an A -module in the obvious way, and a map of A -modules $A \rightarrow M$ is the same datum as an element of M (via evaluation on $1 \in A$). The zero abelian group 0 is an A -module in a unique way, and is both the initial and final object of \mathbf{Mod}_A . A submodule of A is the same as an ideal. There is a natural way of endowing the direct sum or direct product of a family of A -modules with the structure of an A -module. Finite direct sums coincide with finite direct products. The kernel, cokernel, and image of a map of A -modules is an A -module. An A -module is **free** if it is of the form $A^{\oplus S}$ for some set S . A map $\theta : A^{\oplus R} \rightarrow A^{\oplus S}$ between free modules is the same as an $R \times S$ matrix $[\theta_{ij}]_{i \in R, j \in S}$ of elements of A such that for every $i \in R$ we have $\theta_{ij} = 0$ for all but finitely many j . An A -module M is **finitely generated** (or “of finite type”) if there exists a surjection $A^n \rightarrow M$ for some integer $n \geq 0$. A **presentation** of an A -module M is a map between free A -modules $\theta : A^{\oplus R} \rightarrow A^{\oplus S}$ together with an identification $\text{coker}(\theta) \simeq M$. We say that M is **finitely presented** if it admits a presentation with both S and R finite.

A ring A is **Noetherian** if every ideal $I \subseteq A$ is finitely generated. Over a Noetherian ring, every increasing chain of ideals is eventually constant, and the submodule of a finitely generated module is finitely generated. In particular, every finitely generated module is finitely presented. Every quotient A/I and localization $A[S^{-1}]$ of a Noetherian ring A is Noetherian. Moreover, we have the

Theorem B.3.1 (Hilbert’s basis theorem). *Let A be a Noetherian ring and let B be a finitely generated A -algebra. Then B is Noetherian and finitely presented over A .*

In particular, a finitely generated algebra over a field k or over \mathbb{Z} is Noetherian.

A ring is a **principal ideal domain** PID if it is a domain in which every ideal is principal (in particular such a ring is Noetherian). Over a PID, one has the following structure theorem for finitely generated modules.

Theorem B.3.2 (Modules over a PID). *Let A be a PID and let M be a finitely generated A -module. Then*

$$M \simeq A^r \oplus A/(f_1^{n_1}) \oplus \dots \oplus A/(f_m^{n_m})$$

where $r \geq 0$ and $f_1, \dots, f_m \in A$ are prime elements (meaning that each f_i generates a nonzero prime ideal).

For an A -module M and a point $x \in \text{Spec}(A)$, we write $M(x)$ for the base change $M \otimes_A \kappa(x)$ to the residue field. It is a vector space over $\kappa(x)$, called the **fiber** of M at x , which is of finite dimension if M is finitely generated.

Lemma B.3.3 (Nakayama). *Let M be a finitely generated A -module. Then M is zero if and only if $M(x) = 0$ for every closed point $x \in \text{Spec}(A)$.*

Thus if A is a local ring and M is a finitely generated A -module, then $M = 0$ if and only if $M = \mathfrak{m}_A M$. More generally, for a finitely generated M over a local A , elements $m_1, \dots, m_r \in M$ generate M if and only if their images span $M \otimes_A k_A = M/\mathfrak{m}_A M$ as a vector space over $k_A = A/\mathfrak{m}_A$.

A (finite or infinite) sequence of maps of A -modules

$$\dots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \dots$$

is a **complex** if $d^{n-1} \circ d^n = 0$ for all n . Its n -th **cohomology module** is the quotient $H^n = \ker(d^n)/\text{im}(d^{n-1})$. If $H^n = 0$, we say that the sequence is **exact** at the n -th term; if this holds for all n , we say that the complex is exact (or “acyclic”). For example,

$$M \xrightarrow{\alpha} N \longrightarrow Q \longrightarrow 0$$

being exact means that $Q \simeq \text{coker}(\alpha)$, and $M \rightarrow N \rightarrow 0$ is exact if α is surjective. Analogously,

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\alpha} N$$

being exact means $K \simeq \ker(\alpha)$, and $0 \rightarrow M \rightarrow N$ is exact if α is injective. Finally, a **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

which means that M' is a submodule of M and $M'' = M/M'$.

The **tensor product** of two A -modules M and N is an A -module $M \otimes_A N$ determined by the universal property: morphisms of A -modules $M \otimes_A N \rightarrow P$ into an A -module P correspond to A -bilinear maps $M \times N \rightarrow P$. It is generated by the symbols $m \otimes n$ (where $m \in M$ and $n \in N$) subject to the relations

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \quad m \otimes (n + n') = m \otimes n + m \otimes n', \quad xm \otimes n = m \otimes xn$$

for $m, m' \in M, n, n' \in N$, and $x \in A$. We have the relations (or: canonical isomorphisms satisfying certain compatibilities we neglect to mention)

$$(M \otimes N) \otimes P \simeq M \otimes (N \otimes P), \quad M \otimes A = M, \quad M \otimes N \simeq N \otimes M, \quad M \otimes (N \oplus N') = (M \otimes N) \oplus (M \otimes N').$$

The set $\text{Hom}(M, N)$ of all A -module maps $M \rightarrow N$ is an A -module in the obvious way. For A -modules M, N , and P we have a natural isomorphism

$$\text{Hom}(M \otimes_A N, P) \simeq \text{Hom}(M, \text{Hom}(N, P)).$$

Proposition B.3.4 (Tensor product is right exact). *Let $M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules and let N be an A -module. Then the sequence*

$$M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is exact.

This result allows us to describe $M \otimes_A N$ in practice as follows. Let $\theta: A^{\oplus R} \rightarrow A^{\oplus S}$ be a presentation of M , so that $A^{\oplus R} \rightarrow A^{\oplus S} \rightarrow M \rightarrow 0$ is exact. Then

$$N^{\oplus R} \xrightarrow{\theta \otimes N} N^{\oplus S} \longrightarrow M \otimes_A N \longrightarrow 0$$

is exact, where $\alpha \otimes N$ is the map given by “the same matrix” as θ . Thus $M \otimes_A N \simeq \text{coker}(\theta \otimes N)$.

We say that an A -module N is **flat** if tensoring with it preserves short exact sequences (or equivalently, $M' \otimes_A N \rightarrow M \otimes_A N$ is injective if $M' \rightarrow M$ is injective). Free modules are flat. Over a local ring one has the following partial converse, proved using Nakayama’s lemma:

Lemma B.3.5. *A finitely presented flat module over a local ring is free.*

If $\phi^* : A \rightarrow B$ is a map of rings, we can treat B as an A -module. More generally, a B -module can be treated as an A -module via the formula $x \cdot m = \phi^*(x) \cdot m$. This gives a functor $\phi_* : \mathbf{Mod}_B \rightarrow \mathbf{Mod}_A$ (called the **forgetful functor**). In the other direction, if M is an A -module, the tensor product $B \otimes_A M$ has a natural B -module structure, given by $b \cdot (b' \otimes m) = bb' \otimes m$. The module $\phi^*(M) = B \otimes_A M$ is called the **base change** of M to B . This construction gives a (right-exact) functor $\phi^* : \mathbf{Mod}_A \rightarrow \mathbf{Mod}_B$. We have a natural isomorphism, for an A -module M and a B -module N ,

$$\mathrm{Hom}_B(\phi^*M, N) \simeq \mathrm{Hom}_A(M, \phi_*N)$$

(that is, ϕ^* is the left adjoint to ϕ_*).

We say that $\phi^* : A \rightarrow B$ is **flat** if B is flat as an A -module, or equivalently if the base change functor ϕ^* is exact. Crucially, any localization $A \rightarrow A[S^{-1}]$ is flat, and for an A -module M , we have $M \otimes_A A[S^{-1}] = M[S^{-1}]$ (the module of fractions m/s , $m \in M$ and $s \in S$).

Let B and C be A -algebras. Their tensor product $D = B \otimes_A C$ is then an A -algebra with multiplication given by $(b \otimes c)(b' \otimes c') = bb' \otimes cc'$. The maps $B \rightarrow D$, $b \mapsto b \otimes 1$ and $C \rightarrow D$, $c \mapsto 1 \otimes c$ are well-defined A -algebra maps, and D is the categorical coproduct of B and C in the category of A -algebras \mathbf{CAlg}_A . It can be computed in practice as follows. Let $C = A[S]/(R)$ be a presentation of C over A . Then $D = \phi^*(C) \simeq B[S]/(R')$ where R' is the image of R under the map $A[S] \rightarrow B[S]$ induced by the given map $\phi : A \rightarrow B$. Two special cases are of note: if $B = A/I$, then $D = C/IC$, and if $B = A[S^{-1}]$, then $D = C[\phi(S)^{-1}]$. We call D the **base change** of $A \rightarrow B$ to C . We say that a property of morphisms P is stable under base change if $C \rightarrow D$ has P whenever $A \rightarrow B$ has P .

Let $\phi : A \rightarrow B$ be a map of rings and let M be a B -module. An A -linear **derivation** of B into M is an A -module map $\delta : B \rightarrow M$ satisfying the Leibniz rule

$$\delta(xy) = y\delta(x) + x\delta(y).$$

The module of **Kähler differentials** is a B -module $\Omega_{B/A}^1$ together with an A -linear derivation $d : B \rightarrow \Omega_{B/A}^1$ which is universal in the following sense: for every A -linear derivation $\delta : B \rightarrow M$ there exists a unique B -module map $\bar{\delta} : \Omega_{B/A}^1 \rightarrow M$ such that $\delta = \bar{\delta} \circ d$. The module $\Omega_{B/A}^1$ is generated as a B -module by the symbols df ($f \in B$) subject to the rules $d(fg) = fdg + gdf$, $d(f+g) = df + dg$, and $d(\phi^*(a)) = 0$ (for $f, g \in B$ and $a \in A$). Given a presentation $B = A[S]/(R)$ of B over A , we have the following presentation of $\Omega_{B/A}^1$ as a B -module

$$\Omega_{B/A}^1 \simeq \left(\bigoplus_{s \in S} B \cdot dT_s \right) / (df : f \in R)$$

where for $f \in R \subseteq A[S]$, we write $df = \sum_{s \in S} (\partial f / \partial T_s) dT_s$ where $\partial f / \partial T_s$ is the usual formal derivative. In particular, if B is finitely presented over A , then $\Omega_{B/A}^1$ is a finitely presented B -module.

B.4. Integrality and applications

Let $A \rightarrow B$ be a map of rings and let $x \in B$. We say that x is **integral over** A if it satisfies a monic polynomial equation over A :

$$x^n + a_1x^{n-1} + \cdots + a_n = 0, \quad a_1, \dots, a_n \in A.$$

Elements of B which are integral over A form a subring A' of B containing the image of A , called the **integral closure** of A in B . If $A' = A$, we say that A is **integrally closed in** B . If A is a domain, we say that A is **integrally closed** or **normal** if A is integrally closed in $\mathrm{Frac}(A)$.

Theorem B.4.1 (Finiteness of integral closure). *Let A be a domain which is a finitely generated k -algebra, let K be its field of fractions, and let L be a field extension of K of finite degree (possibly $L = K$). Let $B \subseteq L$ be the integral closure of A in L . Then $A \rightarrow B$ is finite. In particular, B is of finite type over k .*

A morphism $A \rightarrow B$ is **integral** if every $x \in B$ is integral over A , and **finite** if B is a finitely generated A -module. A map is finite if and only if it is integral and of finite type. Both finite and integral maps are stable under composition and base change.

Proposition B.4.2 (Going-up). *Let $A \rightarrow B$ be an integral map of rings, then the map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is closed. In particular, if A is a domain and $A \rightarrow B$ is injective, then $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective.*

The following result is extremely useful:

Proposition B.4.3 (Noether normalization lemma). *Let k be a field and let A be a finitely generated k -algebra. Then there exists an $n \geq 0$ and a finite injective k -algebra map*

$$k[T_1, \dots, T_n] \hookrightarrow A.$$

In fact, $n = \dim(A)$ (see below). This lemma is used in some of the proofs of the Nullstellensatz.

Theorem B.4.4 (Essential Nullstellensatz). *Let k be a field and let L be a field extension of k . If L is finitely generated as a k -algebra, then it is a finite extension of k .*

Corollary B.4.5 (Hilbert's Nullstellensatz). *Let k be an algebraically closed field and let A be a finitely generated k -algebra. For every maximal ideal $\mathfrak{m} \subseteq A$, we have $A/\mathfrak{m} = k$ (meaning that $k \rightarrow A \rightarrow A/\mathfrak{m}$ is an isomorphism). This establishes a bijection*

$$\text{MSpec}(A) \simeq \text{Hom}_k(A, k).$$

Let us record an elementary formulation of the Nullstellensatz.

Corollary B.4.6 (Elementary Nullstellensatz). *Let k be an algebraically closed field and let $f_1, \dots, f_r, g \in k[T_1, \dots, T_n]$ be polynomials in n variables. Suppose that for all $(x_1, \dots, x_n) \in k^n$ such that*

$$f_i(x_1, \dots, x_n) = 0 \quad i = 1, \dots, r$$

we have $g(x_1, \dots, x_n) = 0$. Then there exists an integer $m \geq 1$ and polynomials $h_1, \dots, h_r \in k[T_1, \dots, T_n]$ such that $f_1 h_1 + \dots + f_r h_r = g^m$.

The following result of Chevalley describes images of maps between spectra. A subset $W \subseteq X$ of a Noetherian space X is **constructible** if it is a finite union of locally closed subsets.

Theorem B.4.7 (Chevalley). *Let A be a Noetherian ring and let B be an A -algebra of finite type. Then for every constructible subset $W \subseteq \text{Spec}(B)$, the image of W in $\text{Spec}(A)$ is constructible.*

B.5. Dimension theory

The **Krull dimension** $\dim(A)$ of a ring A is the supremum of the set of integers $n \geq 0$ for which there exists a chain of prime ideals $\mathfrak{p}_0 \subseteq \dots \subseteq \mathfrak{p}_n$ of A with $\mathfrak{p}_{i-1} \neq \mathfrak{p}_i$ for all i . If A is a local Noetherian ring or a finitely generated algebra over a field, then $\dim(A)$ is finite.

For a Noetherian local ring A , we have $\dim(A) \leq \dim_{k_A}(\mathfrak{m}_A/\mathfrak{m}_A^2)$. If equality holds, we say that A is **regular**. The following result is not so easy to prove:

Theorem B.5.1. *A regular local ring is a unique factorization domain.*

The **transcendence degree** $\text{trdeg}(K/k)$ of a field extension K/k is the cardinality of any maximal subset $S \subseteq K$ which is algebraically independent over k (meaning that for $x_1, \dots, x_n \in S$ and $f \in k[T_1, \dots, T_n]$ we have $f(x_1, \dots, x_n) \neq 0$). The extension K/k is algebraic if and only if $\text{trdeg}(K/k) = 0$. If K is finitely generated over k , then K is a finite (algebraic) extension of the field of rational functions $k(T_1, \dots, T_n)$ (the fraction field of $k[T_1, \dots, T_n]$) where $n = \text{trdeg}(K/k)$.

Theorem B.5.2. *Let A be a finitely generated domain over a field k and let $K = \text{Frac}(A)$. Then $\dim(A) = \text{trdeg}(K/k)$.*

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