Commutative algebra prerequisites for algebraic geometry January 6, 2025

This document lists the bare definitions and facts we will need for our algebraic geometry course. In case you took *MATH550 Commutative Algebra* in the Fall, this should all be familiar, but can help you with recalling what we have learned. In case you haven't studied some of this material, it should serve as a guideline for what you need to learn.

1. Commutative rings

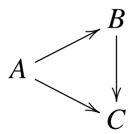
An **commutative ring** (or simply a **ring**) is an abelian group $A = (A, +, 0)$ together with a bilinear associative, commutative, and unital multiplication $\cdot\colon A \times A \to A$, that is: we have an element $1 \in A$ and

$$(x+y)\cdot(x'+y') = xx' + xy' + yx' + yy', \qquad (x\cdot y)\cdot z = x\cdot(y\cdot z), \qquad x\cdot y = y\cdot x, \qquad 1\cdot x = x$$

for all $x, x', y, y', z \in A$. A **homomorphism** (or morphism, or simply map) of rings is a homomorphism of abelian groups $\phi\colon A \to B$ such that $\phi(x\cdot y) = \phi(x)\cdot\phi(y)$ and $\phi(1) = 1$ (the latter condition is not automatic!). Rings form a category which we denote by **CAlg** (for "commutative algebras"). The ring $\mathbb{Z}$ is the initial object of **CAlg**: for every A, there exists a unique map $\mathbb{Z} \to A$. The zero ring 0 is the final object, and if $0 \to A$ is a map, then $A = 0$.

If $\{A_\alpha\}_{\alpha\in I}$ is a family of rings, then the cartesian **product** $A = \prod_{\alpha\in I} A_\alpha$ with coordinate-wise multiplication and addition is a ring, and the projection maps $A \to A_\alpha$ are homomorphisms (A is the categorical product of $\{A_\alpha\}$ in **CAlg**).

For a ring A, an A**-algebra** (or: algebra over A) is a ring B together with a homomorphism $A \to B$ (which we often neglect to name), and a morphism of A-algebras $B \to C$ is a map of rings $B \to C$ such that the triangle



commutes. We denote the category of A-algebras by $\mathbf{CAlg}_A$. Then A (meant as the identity $A \to A$) is the initial object of $\mathbf{CAlg}_A$. We have $\mathbf{CAlg} = \mathbf{CAlg}_{\mathbb{Z}}$.

A **subring** of A is a subgroup $B \subseteq A$ containing 1 and closed under multiplication, or which is the same an injective ring map $B \to A$.

An **ideal** in a ring A is a subgroup $I \subseteq A$ such that $A \cdot I = I$ (that is, if $x \in A$ and $y \in I$, then $xy \in I$). There is a unique ring structure on the quotient group A/I making the quotient map $A \to A/I$ a ring homomorphism. For a map of rings $\phi\colon A \to B$, the kernel $\ker(\phi) \subseteq A$ is an ideal, the image $\operatorname{im}(\phi) \subseteq B$ is a subring, and we have $A/\ker(\phi) \simeq \operatorname{im}(\phi)$. For an ideal $I \subseteq A$, a map $\phi\colon A \to B$ factors (uniquely) through A/I if and only if $I \subseteq \ker(\phi)$. If $\phi\colon A \to B$ is a map and $J \subseteq B$ is an ideal, then $\phi^{-1}(A)$ is an ideal. Ideals in A/I are in bijection with ideals of A containing I. For a subset $I_0 \subseteq A$, the set $I = (I_0)$ of all linear combinations $a_1x_1 + \cdots + a_nx_n$ with $a_i \in A$ and $x_i \in I$ is the smallest ideal of A containing A, called the ideal generated by I_0. For $I_0 = \{x_1, \ldots, x_n\}$, we write $(I_0) = (x_1, \ldots, x_n)$. An ideal of the form $I = (f)$ for a single $f \in A$ is called **principal**. For a map $\phi\colon A \to B$ and an ideal $I \subseteq A$, we write $I \cdot B$ for the ideal of B generated by $\phi(I)$. For a family of ideals $\{I_\alpha\}$, the intersection $\bigcap I_\alpha$ is an ideal, and we denote by $\sum I_\alpha$ the ideal generated by $\bigcup I_\alpha$. For two ideals $I, J \subseteq A$, the product $I \cdot J$ is the ideal generated by $\{xy : x \in I, y \in J\}$.

An ideal $\mathfrak{p} \subseteq A$ is **prime** if its complement $A \setminus \mathfrak{p}$ is a monoid (unital semigroup), or equivalently if $\mathfrak{p} \neq A$ and $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. If $\phi : A \to B$ is a map and $\mathfrak{q} \subseteq B$ is a prime ideal, then $\phi^{-1}(\mathfrak{q}) \subseteq A$ is a prime ideal.

An element $x \in A$ is a **unit** (or invertible) if $xy = 1$ for some $y \in A$. Such a y is unique and is denoted by x^{-1}. Units of a ring A form a group under multiplication, denoted by $A^\times$. A ring A is a **field** if $A^\times = A \setminus \{0\}$ (in particular the zero ring is not a field). A map $A \to B$ induces a map $A^\times \to B^\times$.

A nonzero ring A is a **domain** if it has no zerodivisors, i.e. if $xy = 0$ implies $x = 0$ or $y = 0$. Thus A is a domain if and only if $(0) \subseteq A$ is a prime ideal. An ideal $I \subseteq A$ is prime if and only if A/I is a domain.

An ideal $\mathfrak{m} \subseteq A$ is **maximal** if $\mathfrak{m} \neq A$ and which is maximal with respect to this property. Equivalently, the quotient $A/\mathfrak{m}$ is a field, and every maximal ideal is a prime ideal.

Proposition 1.1. *Every nonzero ring admits a maximal ideal.*

A ring A is **local** if it has a unique maximal ideal $\mathfrak{m}_A$, or equivalently if its non-units form an ideal (which is then maximal). Its **residue field** is the quotient $k_A = A/\mathfrak{m}_A$. A homomorphism $\phi : A \to B$ between local rings is **local** if $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$. It then induces a homomorphism $k_A \to k_B$ between residue fields.

An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n \geq 1$. The set of all nilpotent elements of A is and ideal called its **nilradical** and denoted by $\sqrt{0}$. More generally, for an ideal $I \subseteq A$ the set of all elements $x \in A$ such that $x^n \in I$ for some $n \geq 1$ is an ideal called the **radical** of I and denoted by $\sqrt{I}$. We say that the ideal I is **radical** if $I = \sqrt{I}$. We say that A is **reduced** if it has no nonzero nilpotent elements, i.e. if (0) is a radical ideal.

Proposition 1.2. *For an ideal $I \subseteq A$, we have $\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$ (the intersection of all prime ideals containing I). In particular, an element $x \in A$ is nilpotent if and only if x belongs to every prime ideal of A.*

For a ring A and a (possibly infinite) set S, we denote by $A[S]$ the **polynomial ring** over A in the set of variables T_s for all $s \in S$. We use $A[T_1, \ldots, T_n]$ as a shorthand for $A[\{1, \ldots, n\}]$. This ring has the universal property: for any A-algebra B, giving a map of A-algebras $A[S] \to B$ is the same as giving a map of sets $\gamma \colon S \to B$. An A-algebra B is **finitely generated** (or "of finite type") if there exists an A-algebra surjection $A[S] \to B$ for a finite set S.

For an A-algebra B (i.e. a map $A \to B$), a **presentation** of B over A is a triple (S, γ, R) where S is a set, $\gamma \colon S \to B$ is a map of sets such that the corresponding map $\tilde{\gamma} \colon A[S] \to B$ is surjective, and $R \subseteq A[S]$ is a set which generates $\ker(\tilde{\gamma})$ as an ideal. Thus $B \simeq A[S]/(R)$. We say that B is **finitely presented** over A if it admits a presentation (S, γ, R) where both S and R are finite. If $S = \{1, \ldots, n\}$ and $R = \{f_1, \ldots, f_r\}$, we write

$$B = A[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$$

for the quotient $A[S]/(R)$. Thus a finitely presented A-algebra is one isomorphic to a quotient of the above type. The above quotient has the following universal property: giving an A-algebra map $B \to C$ is the same as giving elements $t_1, \ldots, t_n \in C$ such that $f_i(t_1, \ldots, t_n) = 0$ for $i = 1, \ldots, r$. Here $f_i(t_1, \ldots, t_n)$ is obtained by substituting formally the elements $t_i \in C$ for the variables T_i.

A subset $S \subseteq A$ is a **multiplicative system** if it contains 1 and is closed under multiplication. The **localization** $A[S^{-1}]$ is the ring consisting of fraction symbols a/s where $a \in A$ and $s \in S$, where $a/s = a'/s'$

if $t(as'-a's)=0$ for some $t\in S$, with the usual rules for multiplication and addition. Thus $(1/s)(s/1)=1$ and hence $s/1$ is a unit in $A[S^{-1}]$. The map $A\to A[S^{-1}]$ sending a to $a/1$ is a ring homomorphism and has the following universal property: a map $\phi\colon A\to B$ factors (uniquely) through $A[S^{-1}]$ if and only if $\phi(S)\subseteq B^{\times}$. Key examples:

- For $f\in A$, the set of powers $S=\{1,f,f^2,\ldots\}$ is a multiplicative system and $A[S^{-1}]$ is denoted more succinctly by $A[f^{-1}]$ (we have $A[f^{-1}]\simeq A[T]/(fT-1)$, in particular it is finitely presented over A).
- If $\mathfrak{p}\subseteq A$ is a prime ideal, then $A\setminus\mathfrak{p}$ is a multiplicative system, and we denote $A[(A\setminus\mathfrak{p})^{-1}]$ by $A_{\mathfrak{p}}$ and call it the localization at $\mathfrak{p}$. It is a local ring with unique maximal ideal $\mathfrak{p}\cdot A_{\mathfrak{p}}$.
- In the special case $\mathfrak{p}=(0)$ in a domain A, the ring $A_{(0)}$ is a field, called the **fraction field** of A and denoted by $\mathrm{Frac}(A)$.

In general, the kernel of $A\to A[S^{-1}]$ consists of all $x\in A$ such that $xy=0$ for some $y\in S$, and prime ideals in $A[S^{-1}]$ correspond to prime ideals of A which are disjoint from S. For a prime ideal $\mathfrak{p}\subseteq A$, we denote by $\kappa(\mathfrak{p})$ the residue field of the local ring $A_{\mathfrak{p}}$, or equivalently the fraction field of the quotient $A/\mathfrak{p}$ (we call it the **residue field** of $\mathfrak{p}$).

2. The spectrum

For a ring A, we denote by $\mathrm{Spec}(A)$ the set of all prime ideals of A, called the **spectrum** of A. However, we treat its elements (points) as independent beings, and for $x\in\mathrm{Spec}(A)$ we write $\mathfrak{p}_x$ for the "corresponding prime ideal." For $x\in\mathrm{Spec}(A)$, we denote by $\kappa(x)$ the residue field $\kappa(\mathfrak{p}_x)$, and for $f\in A$ we write $f(x)$ for the image of f in $\kappa(x)$. Thus $f\in A$ defines a "field-valued function" on $\mathrm{Spec}(A)$, where the codomain $\kappa(x)$ depends on the point x. Then $f\in A$ is a unit if and only if $f(x)\neq 0$ for all $x\in\mathrm{Spec}(A)$, and nilpotent if and only if $f(x)=0$ for all $x\in\mathrm{Spec}(A)$.

For $f\in A$, we write

$$D(f)=\{x\in\mathrm{Spec}(A)\,:\,f(x)\neq 0\}\subseteq\mathrm{Spec}(A)$$

(in terms of prime ideals, this is the set of primes containing f). We have $D(fg)=D(f)\cap D(g)$. We give $\mathrm{Spec}(A)$ the topology generated by these sets; thus $Z\subseteq\mathrm{Spec}(A)$ is closed if and only if $Z=V(I)$ for some ideal $I\subseteq A$, where

$$V(I)=\{x\in\mathrm{Spec}(A)\,:\,f(x)=0\text{ for all }f\in I\}\subseteq\mathrm{Spec}(A).$$

The construction $I\mapsto V(I)$ defines a bijection between radical ideals of A and closed subsets of $\mathrm{Spec}(A)$.

The space $\mathrm{Spec}(A)$ is T_0 (for $x\neq y$ we can find an open subset containing exactly one of x,y) but typically not Hausdorff. It is **quasi-compact** (every open cover has a finite subcover), and so are its base open subsets $D(f)\simeq\mathrm{Spec}(A[f^{-1}])$.

An element $x\in A$ is **idempotent** if $x^2=x$; then $y=1-x$ is also idempotent, and $A\simeq A/(x)\times A/(y)$. The space $\mathrm{Spec}(A)$ is the disjoint union of $V(x)=\mathrm{Spec}(A/(x))=D(y)$ and $V(y)=\mathrm{Spec}(A/(y))=D(x)$. Conversely, if $Z\subseteq\mathrm{Spec}(A)$ is a clopen subset, then $Z=V(x)$ for an idempotent x.

On a topological space X, a point x is a **specialization** of a point y (we write $y\rightsquigarrow x$) if x belongs to the closure of $\{y\}$. On $\mathrm{Spec}(A)$, we have $y\rightsquigarrow x$ if and only if $\mathfrak{p}_y\subseteq\mathfrak{p}_x$. In particular, closed points are those whose corresponding prime ideal is maximal. The set of all closed points of $\mathrm{Spec}(A)$ is denoted by $\mathrm{MSpec}(A)$ and called the **maximal spectrum** of A. If A is a domain, we denote by $\eta \in \mathrm{Spec}(A)$ the point corresponding to the prime ideal (0), and call it the **generic point** of $\mathrm{Spec}(A)$. It specializes to every other point.

A ring map $\phi^*\colon A \to B$ induces a continuous map $\phi\colon\ \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ (here we use the geometer's notation, treating Spec as the primary object). In the special case of the quotient $A \to A/I = B$, this map is a homeomorphism of $\mathrm{Spec}(A/I)$ onto $V(I)$. For the map $A \to A[f^{-1}] = B$, the map is a homeomorphism of $\mathrm{Spec}(A[f^{-1}])$ onto $D(f)$. (Thus we think of A/I as "functions on $V(I)$" and of $A[f^{-1}]$ as "functions on $D(f)$" — we will make this precise later using sheaves.) For a domain A, a map $A \to B$ is injective if and only if the generic point $\eta \in \mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$.

For a map $\phi^*\colon A \to B$ and $x \in \mathrm{Spec}(A)$, the **fiber** $\phi^{-1}(x) \subseteq \mathrm{Spec}(B)$ is naturally homeomorphic to $\mathrm{Spec}(B \otimes_A \kappa(x))$ (see the next section for the definition of $\otimes$).

A topological space X is **irreducible** if it is nonempty and cannot be expressed as the union $X = Y_0 \cup Y_1$ of two proper closed subsets $Y_i \subsetneq X$. The spectrum $\mathrm{Spec}(A)$ is irreducible if and only if $\sqrt{0}$ is a prime ideal (for example, if A is a domain), in which case the corresponding point $\eta \in \mathrm{Spec}(A)$ is the unique generic point (meaning that $\{\eta\}$ is dense). More generally, for any A, every irreducible closed subset $Z \subseteq \mathrm{Spec}(A)$ is of the form $\overline{\{x\}}$ for a unique point $x \in \mathrm{Spec}(A)$ (in fact, $Z = V(\mathfrak{p}_x) = \mathrm{Spec}(A/\mathfrak{p}_x)$).

A topological space X is **Noetherian** if every increasing sequence of open subsets is stabilizes, or equivalently if every open subset $U \subseteq X$ is quasi-compact. For example $\mathrm{Spec}(A)$ is Noetherian if A is a Noetherian ring (see below for the definition of a Noetherian ring). A Noetherian topological space can be written uniquely as a finite union of irreducible closed subsets (called its **irreducible components**) $X = Z_1 \cup \ldots \cup Z_r$ such that Z_i is not contained in Z_j for $i \neq j$.

3. Modules

A **module** M over a ring A is an abelian group $(M, 0, +)$ together with a map $\cdot A \times M \to M$ which is unital, bilinear, and associative in the sense that

$$1 \cdot m = m, \qquad (x + x') \cdot (m + m') = x \cdot m + x' \cdot m + x \cdot m' + x' \cdot m', \qquad (x \cdot x') \cdot m = x \cdot (x' \cdot m)$$

for $x, x' \in A$ and $m, m' \in M$ (note that the first $\cdot$ in the third formula denotes multiplication in A). Modules over a field k are precisely the k-vector spaces. A morphism from an A-module M to an A-module N is a homomorphism of abelian groups $\phi\colon M \to N$ such that $\phi(x \cdot m) = x \cdot \phi(m)$. Modules over A form a category denoted by $\mathbf{Mod}_A$.

The ring A is an A-module in the obvious way, and a map of A-modules $A \to M$ is the same datum as an element of M (via evaluation on $1 \in A$). The zero abelian group 0 is an A-module in a unique way, and is both the initial and final object of $\mathbf{Mod}_A$. A submodule of A is the same as an ideal. There is a natural way of endowing the direct sum or direct product of a family of A-modules with the structure of an A-module. Finite direct sums coincide with finite direct products. The kernel, cokernel, and image of a map of A-modules is an A-module. An A-module is **free** if it is of the form $A^{\oplus S}$ for some set S. A map $\theta\colon A^{\oplus R} \to A^{\oplus S}$ between free modules is the same as an $R \times S$ matrix $[\theta_{ij}]_{i \in R, j \in S}$ of elements of A such that for every $i \in R$ we have $\theta_{ij} = 0$ for all but finitely many j. An A-module M is **finitely generated** (or "of finite type") if there exists a surjection $A^n \to M$ for some integer $n \geq 0$. A **presentation** of an A-module

M is a map between free A-modules $\theta \colon A^{\oplus R} \to A^{\oplus S}$ together with an identification $\mathrm{coker}(\theta) \simeq M$. We say that M is **finitely presented** if it admits a presentation with both S and R finite.

A ring A is **Noetherian** if every ideal $I \subseteq A$ is finitely generated. Over a Noetherian ring, every increasing chain of ideals is eventually constant, and the submodule of a finitely generated module is finitely generated. In particular, every finitely generated module is finitely presented. Every quotient A/I and localization $A[S^{-1}]$ of a Noetherian ring A is Noetherian. Moreover, we have the

Theorem 3.1 (Hilbert's basis theorem)**.** *Let A be a Noetherian ring and let B be a finitely generated A-algebra. Then B is Noetherian and finitely presented over A.*

In particular, a finitely generated algebra over a field k or over $\mathbb{Z}$ is Noetherian.

A ring is a **principal ideal domain** PID if it is a domain in which every ideal is principal (in particular such a ring is Noetherian). Over a PID, one has the following structure theorem for finitely generated modules.

Theorem 3.2 (Modules over a PID)**.** *Let A be a PID and let M be a finitely generated A-module. Then*

$$M \simeq A^r \oplus A/(f_1^{n_1}) \oplus \cdots \oplus A/(f_m^{n_m})$$

where $r \geq 0$ and $f_1, \dots, f_m \in A$ are prime elements (meaning that each f_i generates a nonzero prime ideal).

For an A-module M and a point $x \in \mathrm{Spec}(A)$, we write $M(x)$ for the base change $M \otimes_A \kappa(x)$ to the residue field. It is a vector space over $\kappa(x)$, called the **fiber** of M at x, which is of finite dimension if M is finitely generated.

Lemma 3.3 (Nakayama)**.** *Let M be a finitely generated A-module. Then M is zero if and only if $M(x) = 0$ for every closed point $x \in \mathrm{Spec}(A)$.*

Thus if A is a local ring and M is a finitely generated A-module, then $M = 0$ if and only if $M = \mathfrak{m}_A M$. More generally, for a finitely generated M over a local A, elements $m_1, \dots, m_r \in M$ generate M if and only if their images span $M \otimes_A k_A = M/\mathfrak{m}_A M$ as a vector space over $k_A = A/\mathfrak{m}_A$.

A (finite or infinite) sequence of maps of A-modules

$$\cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots$$

is a **complex** if $d^{n-1} \circ d^n = 0$ for all n. Its n-th **cohomology module** is the quotient $H^n = \ker(d^n)/\mathrm{im}(d^{n-1})$. If $H^n = 0$, we say that the sequence is **exact** at the n-th term; if this holds for all n, we say that the complex is exact (or "acyclic"). For example,

$$M \xrightarrow{\alpha} N \longrightarrow Q \longrightarrow 0$$

being exact means that $Q \simeq \mathrm{coker}(\alpha)$, and $M \to N \to 0$ is exact if α is surjective. Analogously,

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\alpha} N$$

being exact means $K \simeq \ker(\alpha)$, and $0 \to M \to N$ is exact if α is injective. Finally, a **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

which means that M' is a submodule of M and $M'' = M/M'$.

The **tensor product** of two A-modules M and N is an A-module $M \otimes_A N$ determined by the universal property: morphisms of A-modules $M \otimes_A N \to P$ into an A-module P correspond to A-bilinear maps $M \times N \to P$. It is generated by the symbols $m \otimes n$ (where $m \in M$ and $n \in N$) subject to the relations

$$(m+m') \otimes n = m \otimes n + m' \otimes n, \qquad m \otimes (n+n') = m \otimes n + m \otimes n', \qquad xm \otimes n = m \otimes xn$$

for $m, m' \in M$, $n, n' \in N$, and $x \in A$. We have the relations (or: canonical isomorphisms satisfying certain compatibilities we neglect to mention)

$$(M \otimes N) \otimes P \simeq M \otimes (N \otimes P), \quad M \otimes A = M, \quad M \otimes N \simeq N \otimes M, \quad M \otimes (N \oplus N') = (M \otimes N) \oplus (M \otimes N').$$

The set $\mathrm{Hom}(M,N)$ of all A-module maps $M \to N$ is an A-module in the obvious way. For A-modules M, N, and P we have a natural isomorphism

$$\mathrm{Hom}(M \otimes_A N, P) \simeq \mathrm{Hom}(M, \mathrm{Hom}(N,P)).$$

Proposition 3.4 (Tensor product is right exact)**.** *Let $M' \to M \to M'' \to 0$ be an exact sequence of A-modules and let N be an A-module. Then the sequence*

$$M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is exact.

This result allows us to describe $M \otimes_A N$ in practice as follows. Let $\theta \colon A^{\oplus R} \to A^{\oplus S}$ be a presentation of M, so that $A^{\oplus R} \to A^{\oplus S} \to M \to 0$ is exact. Then

$$N^{\oplus R} \xrightarrow{\theta \otimes N} N^{\oplus S} \longrightarrow M \otimes_A N \longrightarrow 0$$

is exact, where $\alpha \otimes N$ is the map given by “the same matrix” as θ. Thus $M \otimes_A N \simeq \mathrm{coker}(\theta \otimes N)$.

We say that an A-module N is **flat** if tensoring with it preserves short exact sequences (or equivalently, $M' \otimes_A N \to M \otimes_A N$ is injective if $M' \to M$ is injective). Free modules are flat. Over a local ring one has the following partial converse, proved using Nakayama’s lemma:

Lemma 3.5. *A finitely presented flat module over a local ring is free.*

If $\phi^* \colon A \to B$ is a map of rings, we can treat B as an A-module. More generally, a B-module can be treated as an A-module via the formula $x \cdot m = \phi^*(x) \cdot m$. This gives a functor $\phi_* \colon \mathbf{Mod}_B \to \mathbf{Mod}_A$ (called the **forgetful functor**). In the other direction, if M is an A-module, the tensor product $B \otimes_A M$ has a natural B-module structure, given by $b \cdot (b' \otimes m) = bb' \otimes m$. The module $\phi^*(M) = B \otimes_A M$ is called the **base change** of M to B. This construction gives a (right-exact) functor $\phi^* \colon \mathbf{Mod}_A \to \mathbf{Mod}_B$. We have a natural isomorphism, for an A-module M and a B-module N,

$$\mathrm{Hom}_B(\phi^* M, N) \simeq \mathrm{Hom}_A(M, \phi_* N)$$

(that is, ϕ^* is the left adjoint to ϕ_*).

We say that $\phi^* \colon A \to B$ is **flat** if B is flat as an A-module, or equivalently if the base change functor ϕ^* is exact. Crucially, any localization $A \to A[S^{-1}]$ is flat, and for an A-module M, we have $M \otimes_A A[S^{-1}] = M[S^{-1}]$ (the module of fractions m/s, $m \in M$ and $s \in S$).

Let B and C be A-algebras. Their tensor product $D = B \otimes_A C$ is then an A-algebra with multiplication given by $(b \otimes c)(b' \otimes c') = bb' \otimes cc'$. The maps $B \to D$, $b \mapsto b \otimes 1$ and $C \to D$, $c \mapsto 1 \otimes c$ are well-defined A-algebra maps, and D is the categorical coproduct of B and C in the category of A-algebras $\mathbf{CAlg}_A$. It can be computed in practice as follows. Let $C = A[S]/(R)$ be a presentation of C over A. Then $D = \phi^*(C) \simeq B[S]/(R')$ where R' is the image of R under the map $A[S] \to B[S]$ induced by the given map $\phi \colon A \to B$. Two special cases are of note: if $B = A/I$, then $D = C/IC$, and if $B = A[S^{-1}]$, then $D = C[\phi(S)^{-1}]$. We call D the **base change** of $A \to B$ to C. We say that a property of morphisms P is stable under base change if $C \to D$ has P whenever $A \to B$ has P.

Let $\phi \colon A \to B$ be a map of rings and let M be a B-module. An A-linear **derivation** of B into M is an A-module map $\delta \colon B \to M$ satisfying the Leibniz rule

$$\delta(xy) = y\delta(x) + x\delta(y).$$

The module of **Kähler differentials** is a B-module $\Omega^1_{B/A}$ together with an A-linear derivation $d \colon B \to \Omega^1_{B/A}$ which is universal in the following sense: for every A-linear derivation $\delta \colon B \to M$ there exists a unique B-module map $\bar{\delta} \colon \Omega^1_{B/A} \to M$ such that $\delta = \bar{\delta} \circ d$. The module $\Omega^1_{B/A}$ is generated as a B-module by the symbols df ($f \in B$) subject to the rules $d(fg) = fdg + gdf$, $d(f+g) = df + dg$, and $d(\phi^*(a)) = 0$ (for $f, g \in B$ and $a \in A$). Given a presentation $B = A[S]/(R)$ of B over A, we have the following presentation of $\Omega^1_{B/A}$ as a B-module

$$\Omega^1_{B/A} \simeq \left(\bigoplus_{s \in S} B \cdot dT_s \right) / (df \, : \, f \in R)$$

where for $f \in R \subseteq A[S]$, we write $df = \sum_{s \in S} (\partial f / \partial T_s) dT_s$ where $\partial f / \partial T_s$ is the usual formal derivative. In particular, if B is finitely presented over A, then $\Omega^1_{B/A}$ is a finitely presented B-module.

4. Integrality and applications

Let $A \to B$ be a map of rings and let $x \in B$. We say that x is **integral over** A if it satisfies a monic polynomial equation over A:

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_1, \ldots, a_n \in A.$$

Elements of B which are integral over A form a subring A' of B containing the image of A, called the **integral closure** of A in B. If $A' = A$, we say that A is **integrally closed in** B. If A is a domain, we say that A is **integrally closed** or **normal** if A is integrally closed in $\mathrm{Frac}(A)$.

Theorem 4.1 (Finiteness of integral closure)**.** *Let A be a domain which is a finitely generated k-algebra, let K be its field of fractions, and let L be a field extension of K of finite degree (possibly $L = K$). Let $B \subseteq L$ be the integral closure of A in L. Then $A \to B$ is finite. In particular, B is of finite type over k.*

A morphism $A \to B$ is **integral** if every $x \in B$ is integral over A, and **finite** if B is a finitely generated A-module. A map is finite if and only if it is integral and of finite type. Both finite and integral maps are stable under composition and base change.

Proposition 4.2 (Going-up)**.** *Let $A \to B$ be an integral map of rings, then the map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is closed. In particular, if A is a domain and $A \to B$ is injective, then* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective.*

The following result is extremely useful:

Proposition 4.3 (Noether normalization lemma). *Let k be a field and let A be a finitely generated k-algebra. Then there exists an $n \geq 0$ and a finite injective k-algebra map*

$$k[T_1, \ldots, T_n] \hookrightarrow A.$$

In fact, $n = \dim(A)$ (see below). This lemma is used in some of the proofs of the Nullstellensatz.

Theorem 4.4 (Essential Nullstellensatz). *Let k be a field and let L be a field extension of k. If L is finitely generated as a k-algebra, then it is a finite extension of k.*

Corollary 4.5 (Hilbert's Nullstellensatz). *Let k be an algebraically closed field and let A be a finitely generated k-algebra. For every maximal ideal $\mathfrak{m} \subseteq A$, we have $A/\mathfrak{m} = k$ (meaning that $k \to A \to A/\mathfrak{m}$ is an isomorphism). This establishes a bijection*

$$\mathrm{MSpec}(A) \simeq \mathrm{Hom}_k(A, k).$$

Let us record an elementary formulation of the Nullstellensatz.

Corollary 4.6 (Elementary Nullstellensatz). *Let k be an algebraically closed field and let $f_1, \ldots, f_r, g \in k[T_1, \ldots, T_n]$ be polynomials in n variables. Suppose that for all $(x_1, \ldots, x_n) \in k^n$ such that*

$$f_i(x_1, \ldots, x_n) = 0 \qquad i = 1, \ldots, r$$

we have $g(x_1, \ldots, x_n) = 0$. Then there exists an integer $m \geq 1$ and polynomials $h_1, \ldots, h_r \in k[T_1, \ldots, T_n]$ such that $f_1 h_1 + \cdots + f_r h_r = g^m$.

The following result of Chevalley describes images of maps between spectra. A subset $W \subseteq X$ of a Noetherian space X is **constructible** if it is a finite union of locally closed subsets.

Theorem 4.7 (Chevalley). *Let A be a Noetherian ring and let B be an A-algebra of finite type. Then for every constructible subset $W \subseteq \mathrm{Spec}(B)$, the image of W in $\mathrm{Spec}(A)$ is constructible.*

5. Dimension theory

The **Krull dimension** $\dim(A)$ of a ring A is the supremum of the set of integers $n \geq 0$ for which there exists a chain of prime ideals $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n$ of A with $\mathfrak{p}_{i-1} \neq \mathfrak{p}_i$ for all i. If A is a local Noetherian ring or a finitely generated algebra over a field, then $\dim(A)$ is finite.

For a Noetherian local ring A, we have $\dim(A) \leq \dim_{k_A}(\mathfrak{m}_A/\mathfrak{m}_A^2)$. If equality holds, we say that A is **regular**. The following result is not so easy to prove:

Theorem 5.1. *A regular local ring is a unique factorization domain.*

The **transcendence degree** $\mathrm{trdeg}(K/k)$ of a field extension K/k is the cardinality of any maximal subset $S \subseteq K$ which is algebraically independent over k (meaning that for $x_1, \ldots, x_n \in S$ and $f \in k[T_1, \ldots, T_n]$ we have $f(x_1, \ldots, x_n) \neq 0$). The extension K/k is algebraic if and only if $\mathrm{trdeg}(K/k) = 0$. If K is finitely generated over k, then K is a finite (algebraic) extension of the field of rational functions $k(T_1, \ldots, T_n)$ (the fraction field of $k[T_1, \ldots, T_n]$) where $n = \mathrm{trdeg}(K/k)$.

Theorem 5.2. *Let A be a finitely generated domain over a field k and let $K = \mathrm{Frac}(A)$. Then $\dim(A) = \mathrm{trdeg}(K/k)$.*