MATH551 Algebraic Geometry — Problem Set 1

Due Jan 25, 2026.

Please send your solutions in PDF form to `pachinger@kse.org.ua`. Preferred filename format: `Lastname-N.pdf` where N is the number of the problem set.

In all of the problems, k denotes an algebraically closed field.

Problem 1. Suppose that $\mathrm{char}(k) \neq 2$, let $f \in k[X]$ be a polynomial of degree ≥ 1. Show that the plane curve

$$Y = \{(x,y) : y^2 = f(x)\} \subseteq k^2$$

is irreducible, and that it is smooth if and only if $f(X)$ has no repeated roots. Show that

$$Y = \{(x,y) : y^2 = x^3 + ax + b\} \subseteq k^2$$

is smooth if and only if $4a^3 + 27b^2 \neq 0$.

Problem 2 (The twisted cubic curve, Hartshorne (I, Ex. 1.2))**.** Show that

$$Y = \{(t, t^2, t^3) : t \in k\} \subseteq k^3$$

is an irreducible algebraic set of dimension 1. Find generators for the ideal $\mathcal{I}(Y)$. Show that $k[X,Y,Z]/(\mathcal{I}(Y))$ is isomorphic to a polynomial ring in one variable over k.

Problem 3 (cf. Hartshorne (I, Ex. 1.7))**.** Let X be a Noetherian topological space.

(a) Show that every subset $Z \subseteq X$ is Noetherian in its induced topology.

(b) Show that X is quasi-compact (every open cover has a finite subcover).

(c) Show that if X is also Hausdorff, then it is finite and discrete.

Problem 4 (cf. Hartshorne (I, Ex. 1.10))**.** Let X be a topological space.

(a) If $Y \subseteq X$ is a subspace, then $\dim(Y) \leq \dim(X)$.

(b) Give an example of a topological space X and a dense open subset $U \subseteq X$ with $\dim(U) < \dim(X)$.

(c) Suppose that X is irreducible and $\dim(X) < \infty$. If $Y \subseteq X$ is a closed subset with $\dim(Y) = \dim(X)$, then $Y = X$.

Problem 5 (Hartshorne (I, Ex. 3.7a))**.** Show that every two curves in the projective plane $\mathbb{P}^2$ intersect. In other words, for every pair $f, g \in k[X,Y,Z]$ of non-constant homogeneous polynomials, the system

$$\begin{cases} f(X,Y,Z) = 0, \\ g(X,Y,Z) = 0 \end{cases}$$

has a solution $(x,y,z) \neq (0,0,0)$.

Extra credit

The deadline for all extra credit problems is Apr 13.

$\star$ **Problem 6.** Are the spaces $\mathbb{C}^2$ and $\mathbb{C}^2 \setminus \{(0,0)\}$ homeomorphic when equipped with the Zariski topology?