MATH551 Algebraic Geometry — Problem Set 4

Due Mar 22, 2026.

In this class, we are primarily interested in algebraic sets and varieties. The exercises below are formulated for schemes, but feel free to replace the word *scheme* with *algebraic set* and each Spec with MSpec (the proofs should be the same).

In the problems below, use Problem 1 to solve Problem 2, and Problems 2 and 3 to solve Problem 4.

Recall that an open subset $U \subseteq X$ of an affine scheme X is called **distinguished** if $U = D(f) \simeq \mathrm{Spec}(\mathcal{O}(X)[f^{-1}])$ for some $f \in \mathcal{O}(X)$.

Problem 1. Let X be a scheme. Let $U, V \subseteq X$ be two affine open subsets. Prove that every $x \in U \cap V$ admits an open neighborhood $W \subseteq U \cap V$ which is distinguished in both U and V.

Problem 2 (Affine communication lemma)**.** Let X be a scheme and let $\mathcal{U}$ be a family of affine open subsets of X enjoying the following property: for every affine open $U \subseteq X$ and every finite cover $U = U_1 \cup \cdots \cup U_r$ by distinguished opens $U_1, \ldots, U_r \subseteq U$, we have

$$U \in \mathcal{U} \qquad \Leftrightarrow \qquad U_1, \ldots, U_r \in \mathcal{U}.$$

Suppose that X admits an affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ with $U_\alpha \in \mathcal{U}$. Show that every affine open $U \subseteq X$ belongs to $\mathcal{U}$.

Problem 3. Let X be either a scheme or an algebraic set. Let $f_1, \ldots, f_r \in \mathcal{O}(X)$ be elements such that the opens $D(f_1), \ldots, D(f_r) \subseteq X$ are affine and cover X. Show that X is affine. *Hint:* We have a map $\tau \colon X \to \mathrm{Spec}(\mathcal{O}(X))$ (corresponding to the identity $\mathcal{O}(X) \to \mathcal{O}(X)$). Show that it is an isomorphism.

Problem 4. Let $f \colon Y \to X$ be a map of schemes. Prove that f is affine if and only if there exists an affine open cover $X = \bigcup_{\alpha \in I} U_\alpha$ such that $f^{-1}(U_\alpha)$ is affine for each $\alpha \in I$.

Problem 5. Let X be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Prove that $\mathcal{F}$ is locally free of rank one if and only if there exists a coherent sheaf $\mathcal{M}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M} \simeq \mathcal{O}_X$. Such $\mathcal{O}_X$-modules are called **invertible**.