MATH551 Algebraic Geometry — Problem Set 5

Due Apr 16, 2026.

Problem 1. Let $n \geq 1$. Does there exist a surjective map $\mathbb{A}^n \to \mathbb{P}^n$?

Problem 2. Compute the zeta function $\zeta_{\mathbb{P}^n}(s)$ of the projective space $\mathbb{P}^n$ over a finite field $\mathbb{F}_q$.

Problem 3. Compute $H^1(S^1, \mathbb{Z})$ directly from our definition of cohomology. (Here S^1 is the circle.)

Problem 4. Let X be a noetherian topological space and let $\{\mathcal{F}_\alpha\}$ be an inductive system of abelian sheaves on X. Show that the presheaf $U \mapsto \varinjlim \mathcal{F}_\alpha(U)$ is a sheaf, denoted by $\varinjlim \mathcal{F}_\alpha$. In particular, the natural map

$$\varinjlim \Gamma(X, \mathcal{F}_\alpha) \longrightarrow \Gamma(X, \varinjlim \mathcal{F}_\alpha)$$

is an isomorphism.

Problem 5. Let D be a prime divisor on a smooth variety X. Show that $\mathrm{Pic}(X \setminus D) \simeq \mathrm{Pic}(X)/\langle \mathcal{O}_X(D) \rangle$ where $\langle \mathcal{O}_X(D) \rangle = \{\mathcal{O}_X(nD),\, n \in \mathbb{Z}\}$ is the subgroup generated by the class of $\mathcal{O}_X(D)$. Use this to compute $\mathrm{Pic}(U)$ where $U = D(f) \subseteq \mathbb{P}^2$ is the complement of an irreducible curve of degree d.

$\star$ **Problem 6.** Let $V \simeq \mathbb{R}^n$ be a real vector space endowed with a symmetric bilinear form $\langle x, y \rangle$ of signature $(1, n-1)$. Let $x_1, x_2 \in V$ be two vectors such that

$$\langle x_1, x_1 \rangle = 0 = \langle x_2, x_2 \rangle, \qquad \langle x_1, x_2 \rangle = 1.$$

Show that for every $y_1, y_2 \in V$ the following inequality holds

$$|\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle + \langle x_1, y_2 \rangle \cdot \langle x_2, y_1 \rangle - \langle y_1, y_2 \rangle| \leq \sqrt{2\langle x_1, y_1 \rangle \cdot \langle x_2, y_1 \rangle - \langle y_1, y_1 \rangle} \cdot \sqrt{2\langle x_1, y_2 \rangle \cdot \langle x_2, y_2 \rangle - \langle y_2, y_2 \rangle}.$$

Hint: Apply Cauchy–Schwarz to the quadratic form $q(y) = 2\langle x_1, y \rangle \cdot \langle x_2, y \rangle - \langle y, y \rangle$.