# *Problem Set 1*

*due March 13, 2025*

**Problem 1.** Prove that the free group $F_n$ on $n \geq 1$ generators is not isomorphic to the fundamental group of a smooth projective complex variety.

*Hint:* Reduce to the case $n$ odd by passing to a finite index subgroup.

**Problem 2.** Let $\Gamma$ be a discrete group. Denote by $\mathcal{C}_\Gamma$ the category of sets endowed with a $\Gamma$-action, and by $F\colon \mathcal{C}_\Gamma \to \mathrm{Set}$ the forgetful functor. Prove that $\Gamma$ is canonically isomorphic to the automorphism group $\mathrm{Aut}(F)$ of the functor $F$.

**Problem 3.** A **profinite group** is a topological group which is isomorphic (as a topological group) to the projective limit $\varprojlim G_i$ of an inverse system $(G_i)_{i\in I}$ of finite groups (with the discrete topology). We will learn quite a bit about them later, here are some warm-ups.

(a) Let $\Gamma$ be a group and let $H \subseteq \Gamma$ be a subgroup of finite index. Prove that there exists a normal subgroup $N \subseteq \Gamma$ of finite index which is contained in $H$.

(b) Prove that a topological group is profinite if and only if it is compact Hausdorff and totally disconnected.

(c) Let $\Gamma$ be a group. Prove that there exists an initial homomorphism $\Gamma \to \widehat{\Gamma}$ into a profinite group, called the **profinite completion** of $\Gamma$. Identify $\widehat{\Gamma}$ with the inverse limit of $\Gamma/H$ over all normal finite index subgroup $H \subseteq \Gamma$, and establish a bijection between open (normal) subgroups of $\widehat{\Gamma}$ and finite index (normal) subgroups of $\Gamma$.

**Problem 4.** Let $X$ be a topological space and let $G$ be an abelian group. Construct a bijection between $H^1(X,G)$ and the set of isomorphism classes of $G$-torsors on $X$.

**Problem 5.** Let $\ell_0, \ell_1, \ell_2 \subseteq \mathbb{C}^2$ be three lines in general position (no two are parallel, and they do not all pass through one point), and let

*Hint:* Find the computation in the literature if you are stuck (you still need to provide a complete proof).

$$X = \mathbb{C}^2 \setminus (\ell_0 \cup \ell_1 \cup \ell_2).$$

Compute the fundamental group of $X$. Is $X$ a $K(\pi,1)$?