Hodge theory over $\mathbf{C}((t))$

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Piotr Achinger

IMPAN Warsaw

I

Complex and Kähler geometry

Complex manifolds

What can we say about the homotopy type of a complex manifold X?

$$H^*(X,\mathbf{Z}), \quad \pi_1(X), \quad \ldots$$

Example questions/results:

- Can X be $\mathbf{S}^{2n}$? Open for $n=3$.
- $\pi_1(X)$ can be any finitely presented group (Taubes)

We can say more if X is Kähler.

Kähler manifolds

X compact Kähler manifold (e.g. $X \hookrightarrow \mathbf{P}^n$)

1. $H^n(X, \mathbf{Q})$ carries a Hodge structure

$$H^n(X,\mathbf{Q}) \otimes \mathbf{C} \simeq \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = H^q(X, \Omega_X^p), \quad \overline{H^{p,q}} = H^{q,p}.$$

2. Hard Lefschetz:

$$- \cup [\omega]^k \quad : \quad H^{d-k}(X,\mathbf{C}) \xrightarrow{\sim} H^{d+k}(X,\mathbf{C}).$$

3. nonabelian Hodge theory $\Rightarrow$ restrictions on $\pi_1(X)$ (Kähler groups)

Example. Hopf surface

$$X = (\mathbf{C}^2 \setminus 0)/q^{\mathbf{Z}}, \quad 0 < |q| < 1$$

It is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^3$. Thus $\pi_1(X) \simeq \mathbf{Z}$ and X is not Kähler.

II

Non-Archimedean geometry

Rigid-analytic varieties

K non-Archimedean field e.g. $k((t))$, $\mathbf{Q}_p$, $\mathbf{C}_p$

Tate: theory of rigid-analytic varieties over K

1. Tate algebra

$$K\langle x_1, \ldots, x_r \rangle = \left\{ \sum_{n \in \mathbf{N}^r} a_n \mathbf{x}^n \in K[[x_1, \ldots, x_r]] : a_n \to 0 \text{ as } |n| \to \infty \right\}$$

2. Affinoid spaces

$$X = \operatorname{Sp} K\langle x_1, \ldots, x_r \rangle / I$$

(underlying set = maximal ideals)

3. Glued together using the admissible topology

Rigid-analytic varieties = generic fibers of formal schemes

$\mathcal{O} = \{|x| \leq 1\} \subseteq K$ valuation ring $\qquad 0 < |t| < 1$ pseudouniformizer

$$K = \left(\varprojlim_n \mathcal{O}/t^{n+1}\right)\left[\frac{1}{t}\right]$$

$$K\langle X_1, \ldots, X_r\rangle = \left(\varprojlim_n \mathcal{O}[X_1, \ldots, X_r]/t^{n+1}\right)\left[\frac{1}{t}\right]$$

$$\{\text{Rigid varieties}/K\} \overset{?}{\simeq} \left(\varprojlim_n \mathbf{Sch}^{\text{f.t.}}_{\mathcal{O}/t^{n+1}}\right)\left[\frac{1}{t}\right]$$

Theorem (Raynaud, Bosch–Lütkebohmert)

$$\left\{\begin{matrix}\text{qcqs rigid-analytic} \\ \text{varieties over } K\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{admissible formal} \\ \text{schemes over } \mathcal{O}\end{matrix}\right\}[\text{admissible blowups}^{-1}]$$

Semistable models

Setup:

- $K = \mathbf{C}(\!(t)\!)$ from now on, so $\mathcal{O} = \mathbf{C}[\![t]\!]$
- $\mathfrak{X} = (X_n / \operatorname{Spec} \mathcal{O}/t^{n+1})$ semistable formal scheme
- $Y = X_0/\mathbf{C}$ with induced log structure log special fiber
- $X = \mathfrak{X}_K$ rigid-analytic generic fiber; $\mathfrak{X}$ is called a formal model of X

$$\begin{array}{ccccc} X & \longrightarrow & \mathfrak{X} & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Sp} K & \longrightarrow & \operatorname{Spf} \mathcal{O} & \longleftarrow & \operatorname{Spec} \mathbf{C} \end{array}$$

Theorem (Semistable reduction)

Every smooth qcqs rigid-analytic variety X over $K = \mathbf{C}(\!(t)\!)$ admits a semistable formal model $\mathfrak{X}$ over $\mathbf{C}[\![t^{1/m}]\!]$ for some $m \geq 1$.

III

The Betti homotopy type

Log schemes simplified

Working definition

A *(DF) log structure* on a scheme X is a tuple of maps from line bundles to $\mathcal{O}_X$

$$s_i : L_i \to \mathcal{O}_X, \quad i = 1, \ldots, s.$$

Examples.

1. X regular, $Y = Y_1 \cup \ldots \cup Y_s \subseteq X$ an snc divisor
   $\rightsquigarrow$ log structure $\{s_i = \mathcal{O}_X(-Y_i) \to \mathcal{O}_X\}$

   E.g. $X = \operatorname{Spec} \mathbf{C}[t]$ or $\operatorname{Spec} \mathbf{C}[[t]]$ and $Y = \{t = 0\}$, $\{t : \mathcal{O}_X \to \mathcal{O}_X\}$

2. Restrict the log structure to Y: $\{s_i = \mathcal{O}_Y(-Y_i)|_Y \to \mathcal{O}_Y\}$

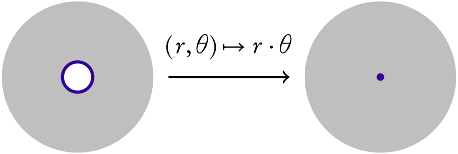
   E.g. $Y = \operatorname{Spec} \mathbf{C}$ with $\{0 : \mathcal{O}_Y \to \mathcal{O}_Y\}$ standard log point

3. This applies also to $Y = \mathcal{X}_0$ the special fiber of our semistable formal scheme $\mathcal{X}$ over $\mathcal{O}$.

Kato–Nakayama space $X_{\log}$ of a log scheme X

Functor $X \mapsto X_{\log}$: {f.t. log schemes/$\mathbf{C}$} $\to$ {topological spaces}, modeled on

$\overline{\mathbf{C}} = \mathbf{R}_{\geq 0} \times \mathbf{S}^1$



$\mathbf{C}$

X a scheme of finite type over $\mathbf{C}$ with a log structure

$$s_i : L_i \to \mathcal{O}_X \quad \longleftrightarrow \quad \sigma_i : X \to [\mathbf{A}^1/\mathbf{G}_m].$$

We take the pull-back

$$\begin{tikzcd}
X_{\log} \arrow[rr] \arrow[d, "\tau"'] \arrow[drr, phantom, "\square"] & & {[\overline{\mathbf{C}}/\mathbf{C}^\times]^s} \arrow[d] \\
X_{\mathrm{an}} \arrow[r, "\prod \sigma_i"'] & \prod_{i=1}^s [\mathbf{A}^1/\mathbf{G}_m]_{\mathrm{an}} \arrow[r, equal] & {[\mathbf{C}/\mathbf{C}^\times]^s}
\end{tikzcd}$$

E.g. For the standard log point ($\operatorname{Spec} \mathbf{C}, 0 \colon \mathcal{O} \to \mathcal{O}$) we have $X_{\log} = \mathbf{S}^1$.

The Kato–Nakayama space of the special fiber Y

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & Y \arrow[l] \arrow[d] & Y_{\log} \arrow[l, dotted] \arrow[d, dotted] \\
\operatorname{Sp} K \arrow[r] & \operatorname{Spf} \mathcal{O} & \operatorname{Spec} \mathbf{C} \arrow[l] & \mathbf{S}^1 \arrow[l, dotted]
\end{tikzcd}
$$

Slogan: the topology $Y_{\log}$ reflects the topology of X with its monodromy

Theorem (Nakayama–Ogus)

1. The space $Y_{\log}$ is a manifold with corners.
2. If Y is proper, then $Y_{\log} \to \mathbf{S}^1$ is proper and a locally trivial fibration.

The Betti homotopy type

Theorem (A.–Talpo)

The homotopy type of $Y_{\log}/\mathbf{S}^1$ does not depend on the choice of $\mathfrak{X}$.

This gives rise to a functor

$$\Psi_{\mathrm{rig}} \colon \{\text{smooth rigid-analytic spaces over } K\} \longrightarrow (\infty\text{-category of spaces}/\mathbf{S}^1).$$

Theorem (Stewart–Vologodsky, Berkovich)

The cohomology groups

$$H^*(\widetilde{\Psi}(X), \mathbf{Z}) := H^*(\widetilde{Y}_{\log}, \mathbf{Z}), \quad \widetilde{Y}_{\log} = Y_{\log} \times_{\mathbf{S}^1} \mathbf{R}(1)$$

carry a natural MHS.

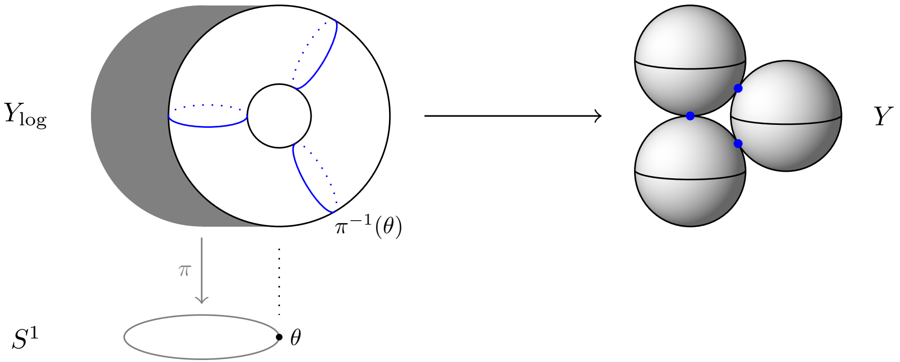
Examples

Example 1. DWORK ELLIPTIC CURVE

$$X = \{\, t(X^3+Y^3+Z^3) = XYZ \,\} \subseteq \mathbf{P}^2_{\mathbf{C}((t))}$$

$$\mathcal{X} = \{\, t(X^3+Y^3+Z^3) = XYZ \,\} \subseteq \mathbf{P}^2_{\mathbf{C}[[t]]}$$

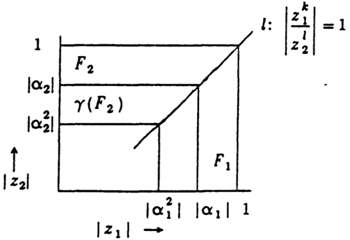
$$Y = \{\, 0 = XYZ \,\} \subseteq \mathbf{P}^2_{\mathbf{C}}$$



Examples

Example 2. NON-ARCHIMEDEAN HOPF SURFACE

$$X = (\mathbf{A}^2_{\mathbf{C}((t))} \setminus 0)_{\mathrm{an}} / t^{\mathbf{Z}}$$



(source: H. Voskuil *Non archimedean Hopf surfaces* 1991)

Special fiber $Y = \mathrm{Bl}_P\, \mathbf{P}^2_{\mathbf{C}} / (\tilde{L} \sim E)$.

We have $\Psi_{\mathrm{rig}}(X) \simeq \mathbf{S}^1 \times \mathbf{S}^3$. **Coincidence?**

IV

Projective reduction and Hodge symmetry

Rigid varieties with projective reduction

Definition (Li)

A rigid variety X over K has *projective reduction* if there exists a formal model $\mathcal{X}$ with $Y = \mathcal{X}_0$ projective.

Intuition: projective reduction $\sim$ Kähler

Theorem (Li, Hansen–Li)

If X has projective reduction, then $\mathrm{Pic}^0(X)$ is proper and $h^{1,0}(X) = h^{0,1}(X)$.

They **asked** whether then $h^{p,q}(X) = h^{q,p}(X)$ for all $p, q \geq 0$.

This was **disproved** by Petrov (2020): he found $\mathcal{X}$ over $\mathbf{Z}_p$ with good reduction and $h^{3,0}(X) \neq h^{0,3}(X)$.

Hodge symmetry over $\mathbf{C}(\!(t)\!)$

Theorem (A. 2020)

If $X/\mathbf{C}(\!(t)\!)$ is a smooth and proper rigid-analytic space with projective reduction, then $h^{p,q}(X) = h^{q,p}(X)$ for $p, q \geq 0$.

Proof steps:

1. (Nakkajima) The weight-monodromy thm implies log hard Lefschetz

$$c_1(L)^k : H^{d-k}(Y, \Omega^{\bullet}_{Y/\mathbf{C}}(\log)) \xrightarrow{\sim} H^{d+k}(Y, \Omega^{\bullet}_{Y/\mathbf{C}}(\log)).$$

This is compatible with the Hodge filtrations, showing

$$\dim H^p(Y, \Omega^q_{Y/\mathbf{C}}(\log)) = \dim H^{d-q}(Y, \Omega^{d-p}_{Y/\mathbf{C}}(\log)).$$

By log Serre duality (Tsuji), the latter equals $\dim H^q(Y, \Omega^p_{Y/\mathbf{C}}(\log))$.

2. (Illusie–Kato–Nakayama) The relative log Hodge cohomology $H^q(\mathcal{X}, \Omega^p_{\mathcal{X}/\mathcal{O}}(\log))$ is locally free and commutes with base change.

V

The Riemann–Hilbert correspondence

Classical Riemann–Hilbert correspondence

For a complex manifold X, one has the equivalence

$$\mathrm{LocSys}_{\mathbf{C}}(X) \simeq \mathrm{MIC}(X)$$

between $\mathbf{C}$-local systems and holomorphic vector bundles with an integrable connection.

Theorem (Deligne's Riemann–Hilbert correspondence)

For a smooth complex algebraic variety X, one has an equivalence

$$\mathrm{LocSys}_{\mathbf{C}}(X_{\mathrm{an}}) \simeq \mathrm{MIC}_{\mathrm{reg}}(X),$$

where $\mathrm{MIC}_{\mathrm{reg}}(X)$ denotes algebraic vector bundles with an integrable connection which are regular at infinity.

Riemann–Hilbert on rigid-analytic spaces

$\mathrm{MIC}(X/\mathbf{C}) = \{\mathbf{C}\text{-linear int. conn. on } X\}$ (so $\tau = t\frac{d}{dt}$ acts)

$\mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}) \subseteq \mathrm{MIC}(X/\mathbf{C})$ *regular* connections

$\mathrm{LocSys}_{\mathbf{C}}(\Psi_{\mathrm{rig}}(X)) = \mathbf{C}\text{-local systems on } Y_{\log}$ (indep. of model $\mathcal{X}$)

"Theorem" (Riemann–Hilbert for rigid-analytic spaces)

Let X be a smooth qcqs rigid-analytic space over $K = \mathbf{C}((t))$. There is an equivalence of categories

$$\mathrm{LocSys}_{\mathbf{C}}(\Psi_{\mathrm{rig}}(X)) \simeq \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C}).$$

Proof relies on a similar theorem for regular log connections on Y.

VMHS on rigid-analytic spaces

Definition (tentative)

A *variation of mixed Hodge structure* (VMHS) on X consists of

- $V \in \mathrm{MIC}_{\mathrm{reg}}(X/\mathbf{C})$ with a Griffiths-transverse Hodge filtration $F^\bullet V$,
- $\mathcal{V} \in \mathrm{LocSys}_{\mathbf{Q}}(\Psi_{\mathrm{rig}}(X))$ with a weight filtration $W_\bullet \mathcal{V}$,
- an isomorphism $\iota \colon \mathrm{RH}(V) \simeq \mathcal{V}_{\mathbf{C}}$,

such that for every classical point $s \colon \operatorname{Sp} \mathbf{C}(\!(t^{1/N})\!) \to X$, the pull-back

$$s^*(V, F^\bullet, \mathcal{V}, W_\bullet, \iota)$$

is an “admissible limit VMHS.”

VI

Open questions

Some open questions

1. What are the possible $\pi_1(\Psi_{\mathrm{rig}}(X))$?

compact complex manifolds	compact Kähler manifolds	proper smooth rigid varieties over $\mathbf{C}((t))$	…with projective reduction
$\pi_1(X)$=any f.p. group (Taubes)	Kähler groups	can $\pi_1(\Psi_{\mathrm{rig}}(X))$ be any f.p. gp?	can $\pi_1(\Psi_{\mathrm{rig}}(X))$ be non-Kähler?

2. Can $\Psi_{\mathrm{rig}}(X)$ be homotopy equivalent to S^{2n}?

3. Link between the rigid motive of X (Ayoub) and the log motive of Y (Binda–Park–Østvaer)?